Efficiency

Announcements

Measuring Efficiency

Recursive Computation of the Fibonacci Sequence

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fib(5)

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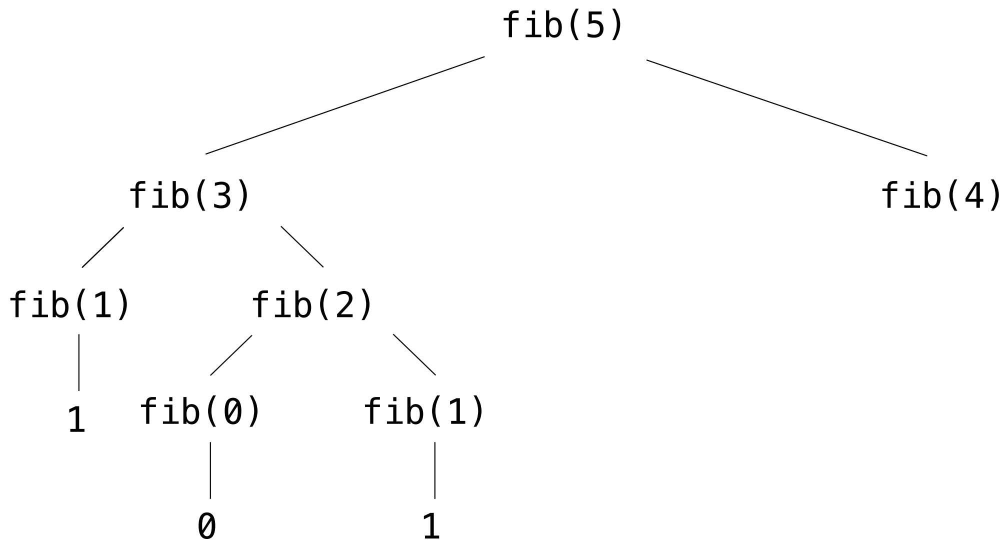


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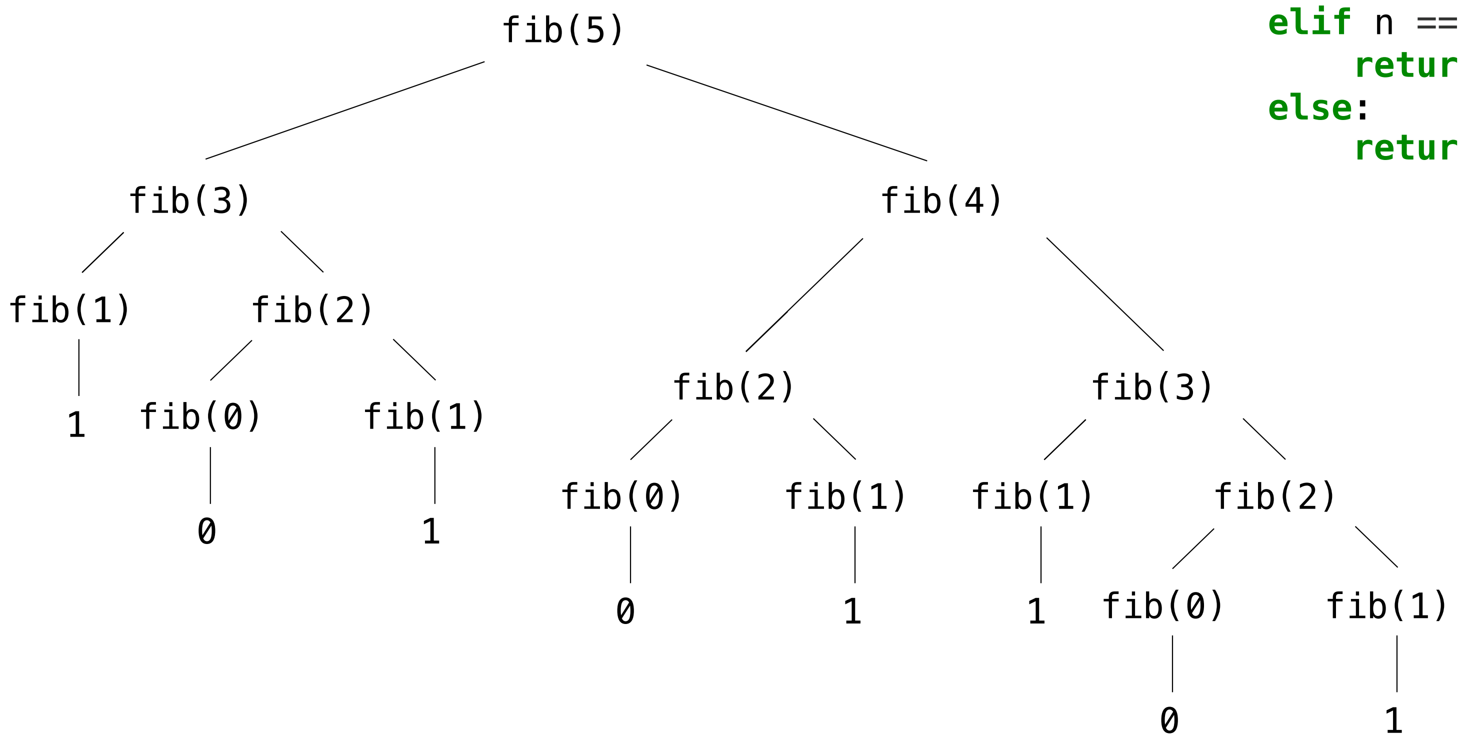

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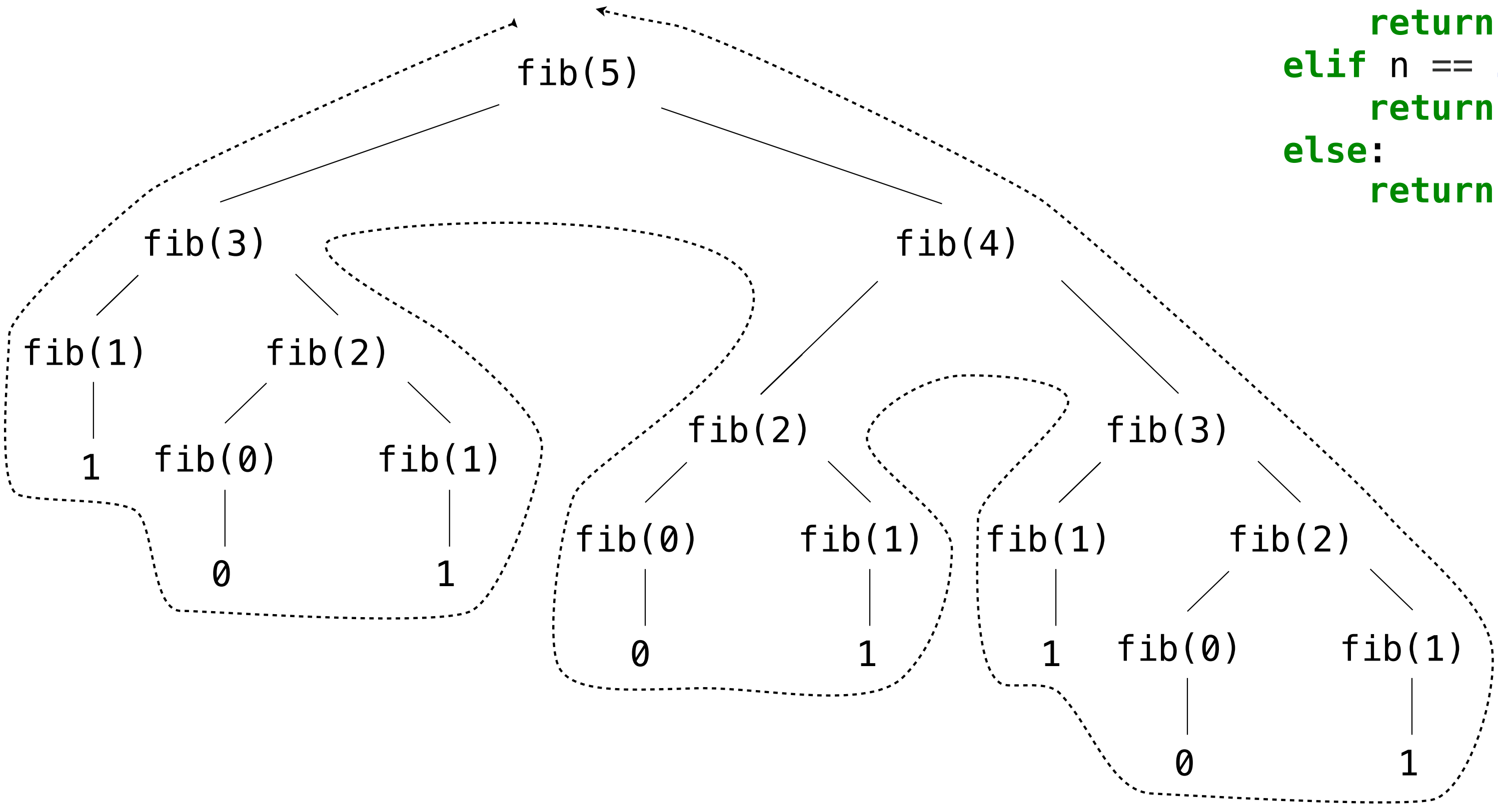

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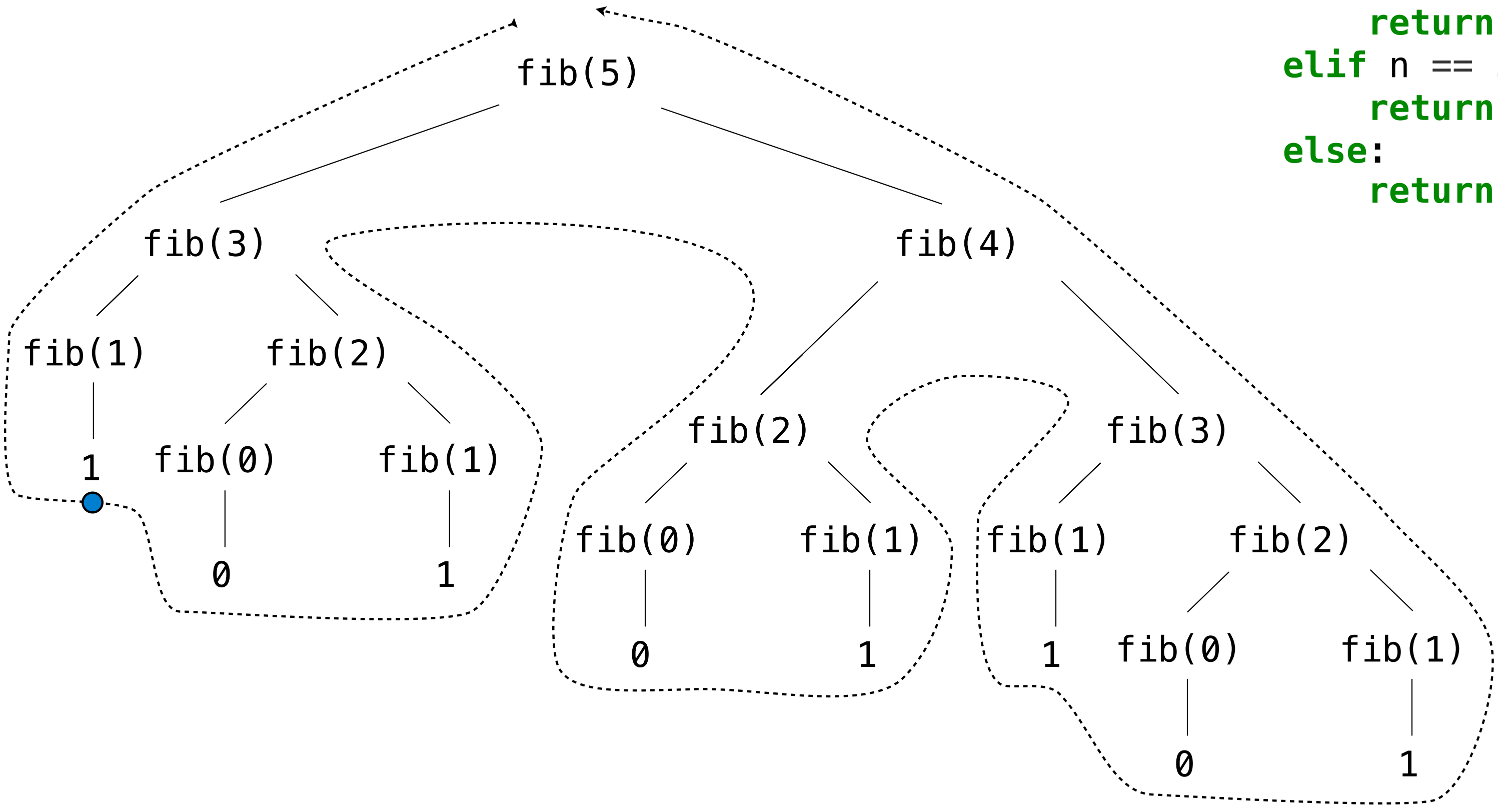

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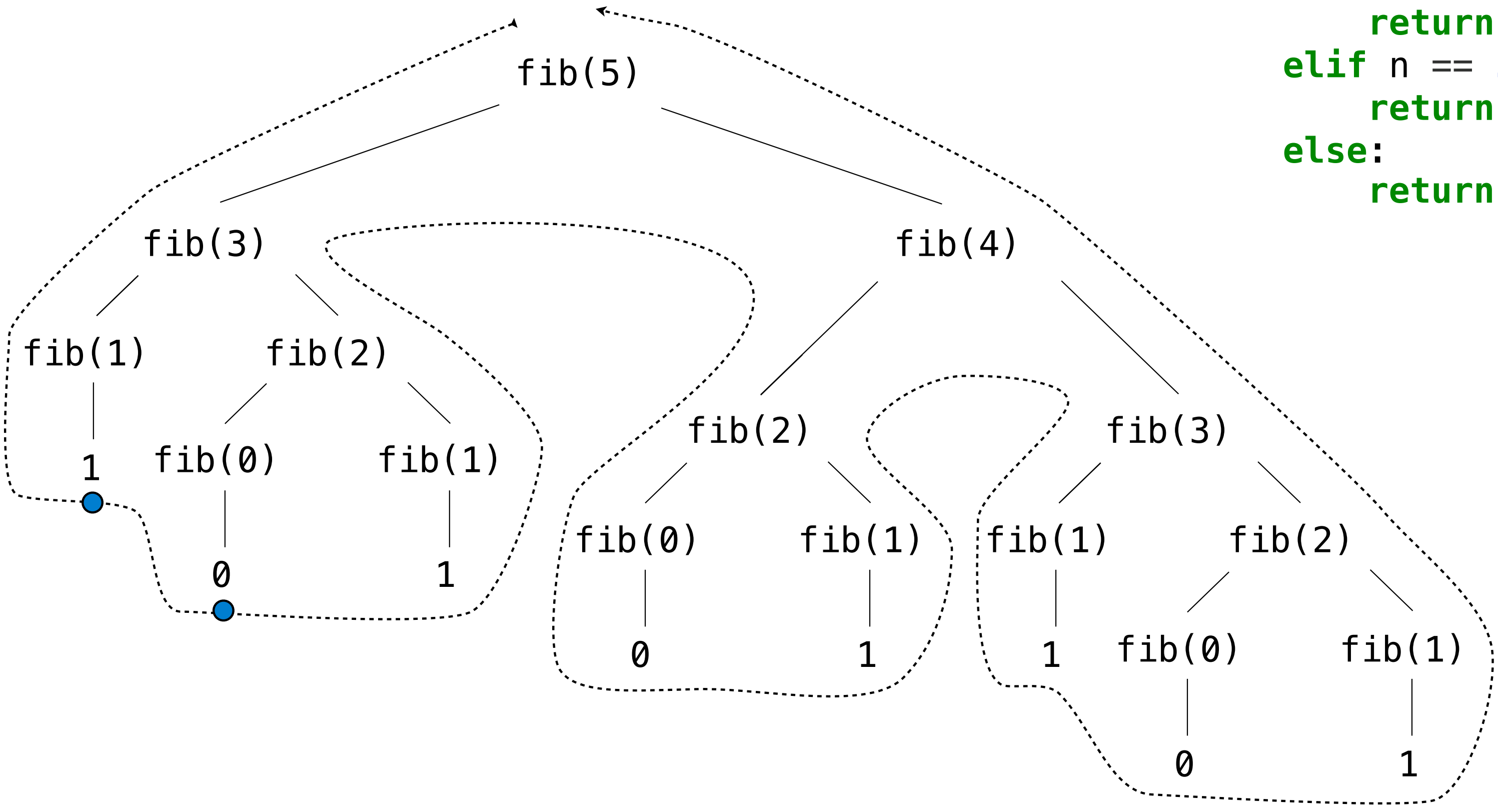

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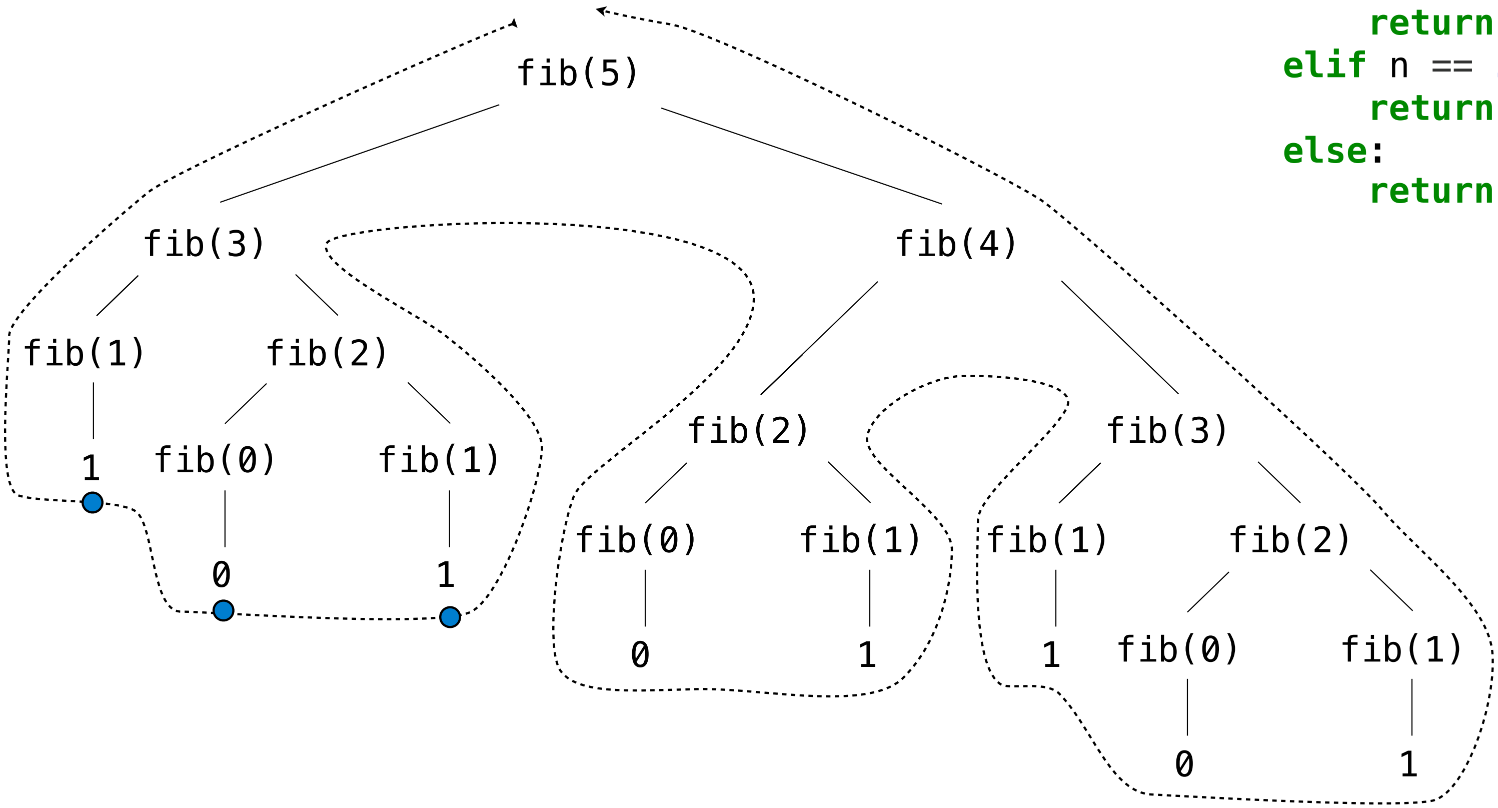

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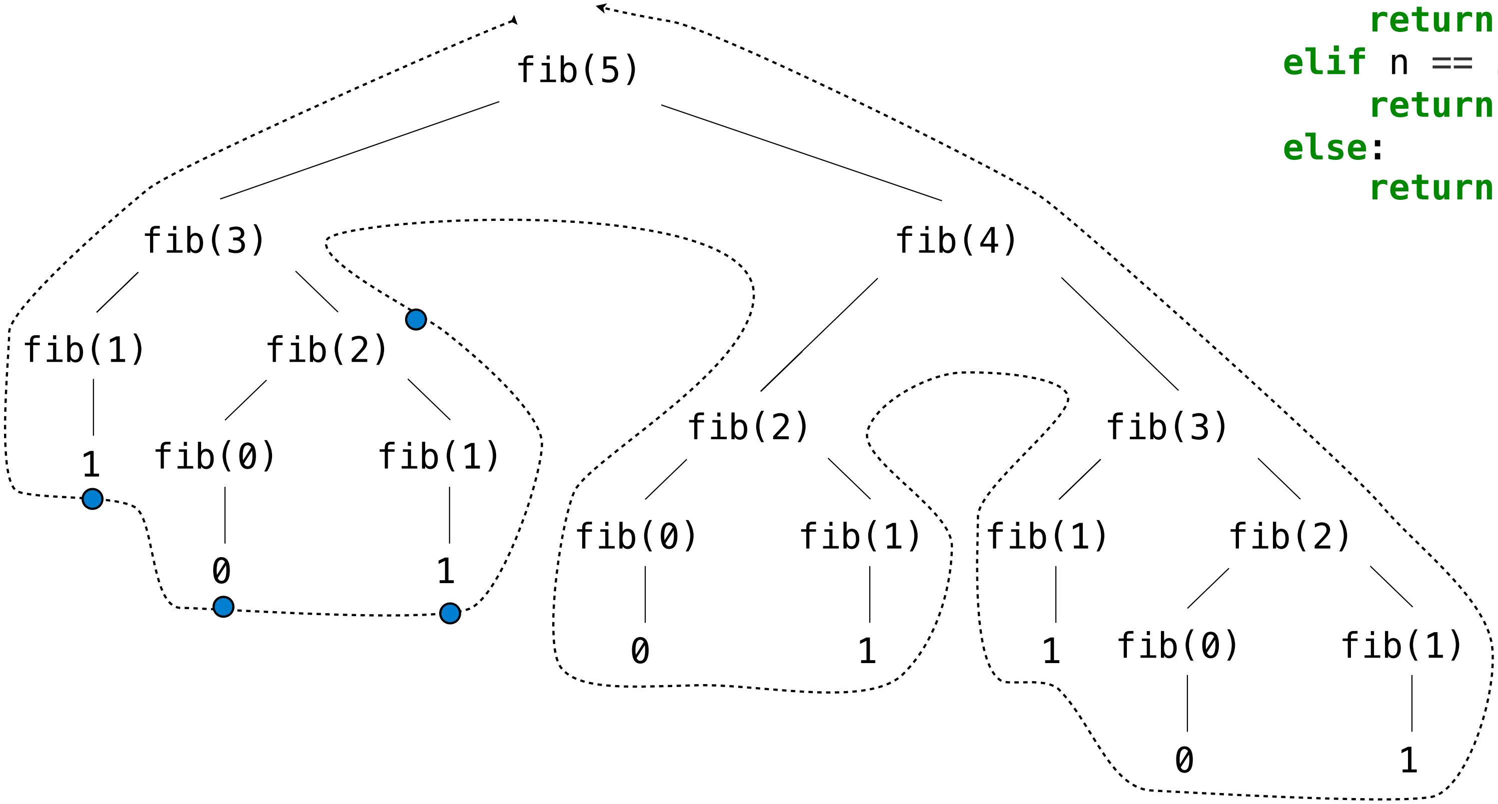

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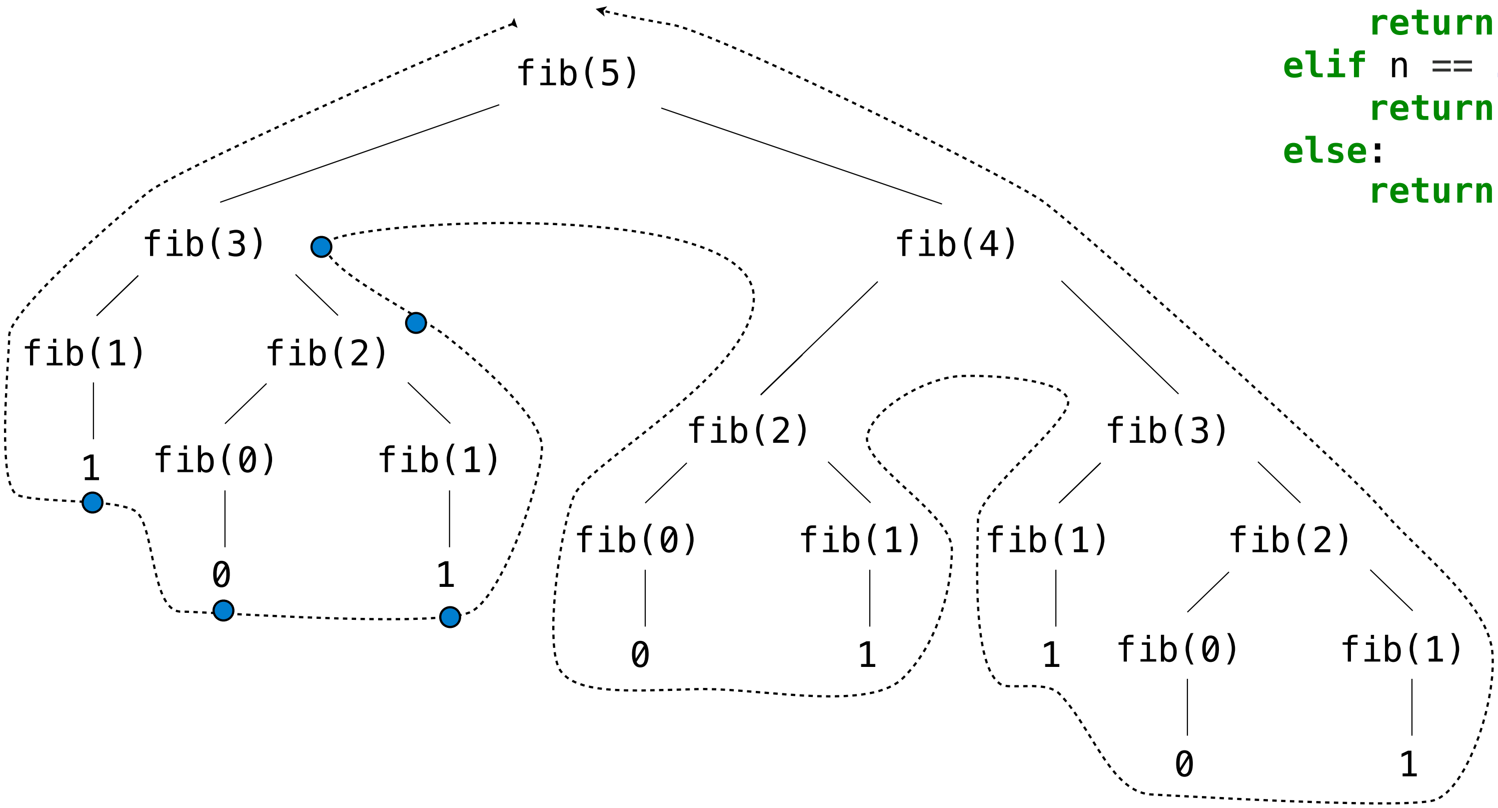

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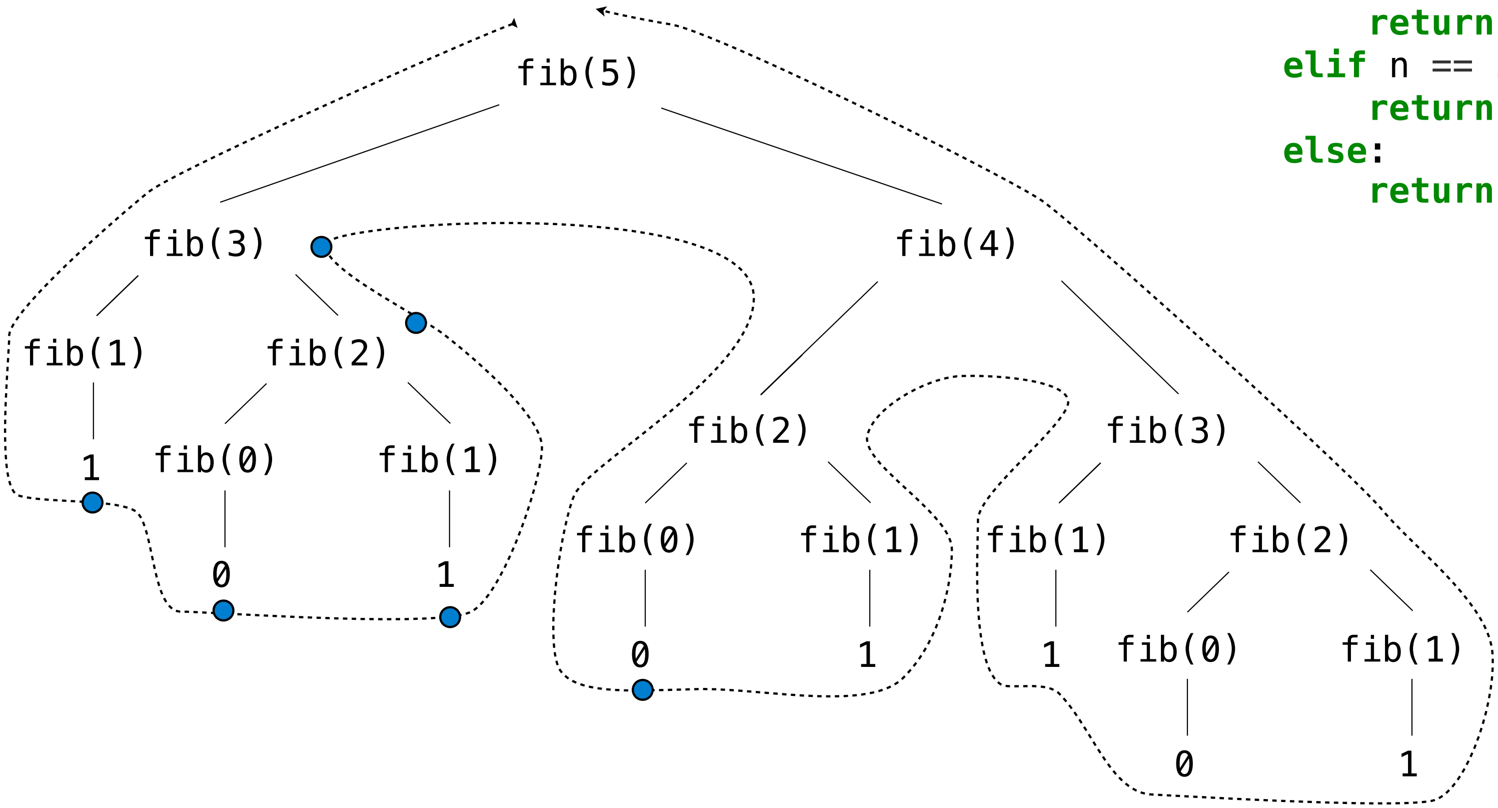

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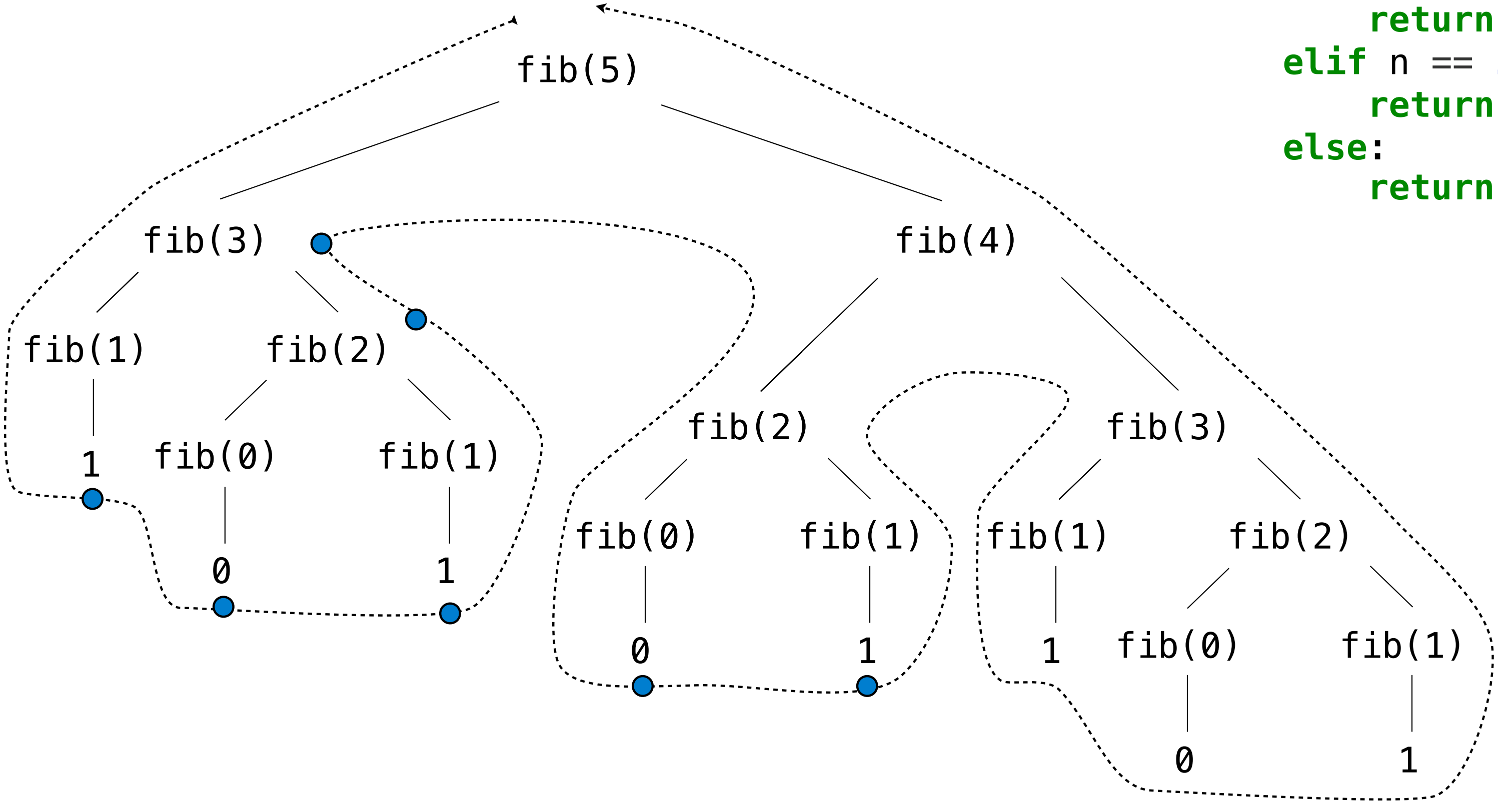


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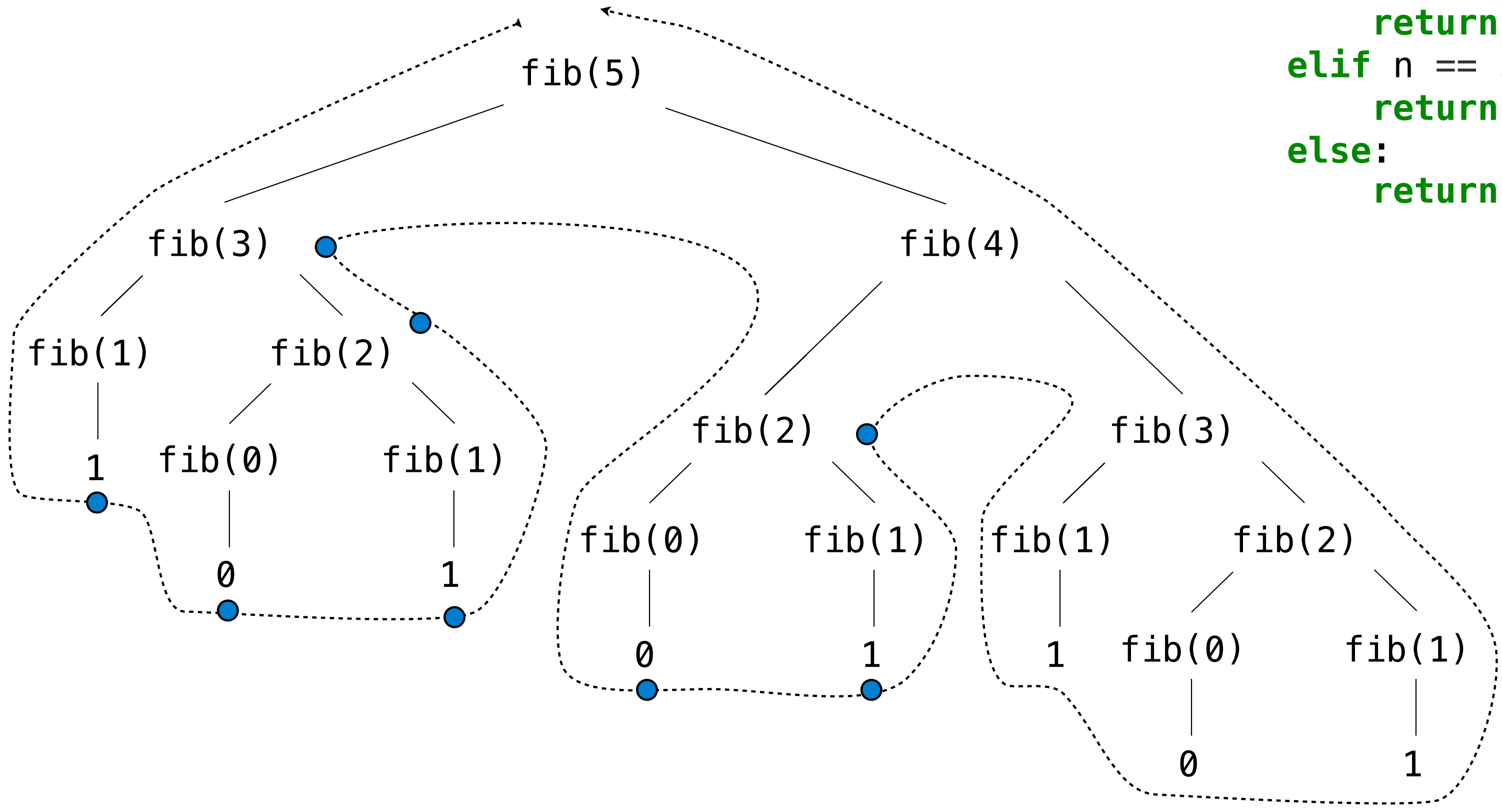

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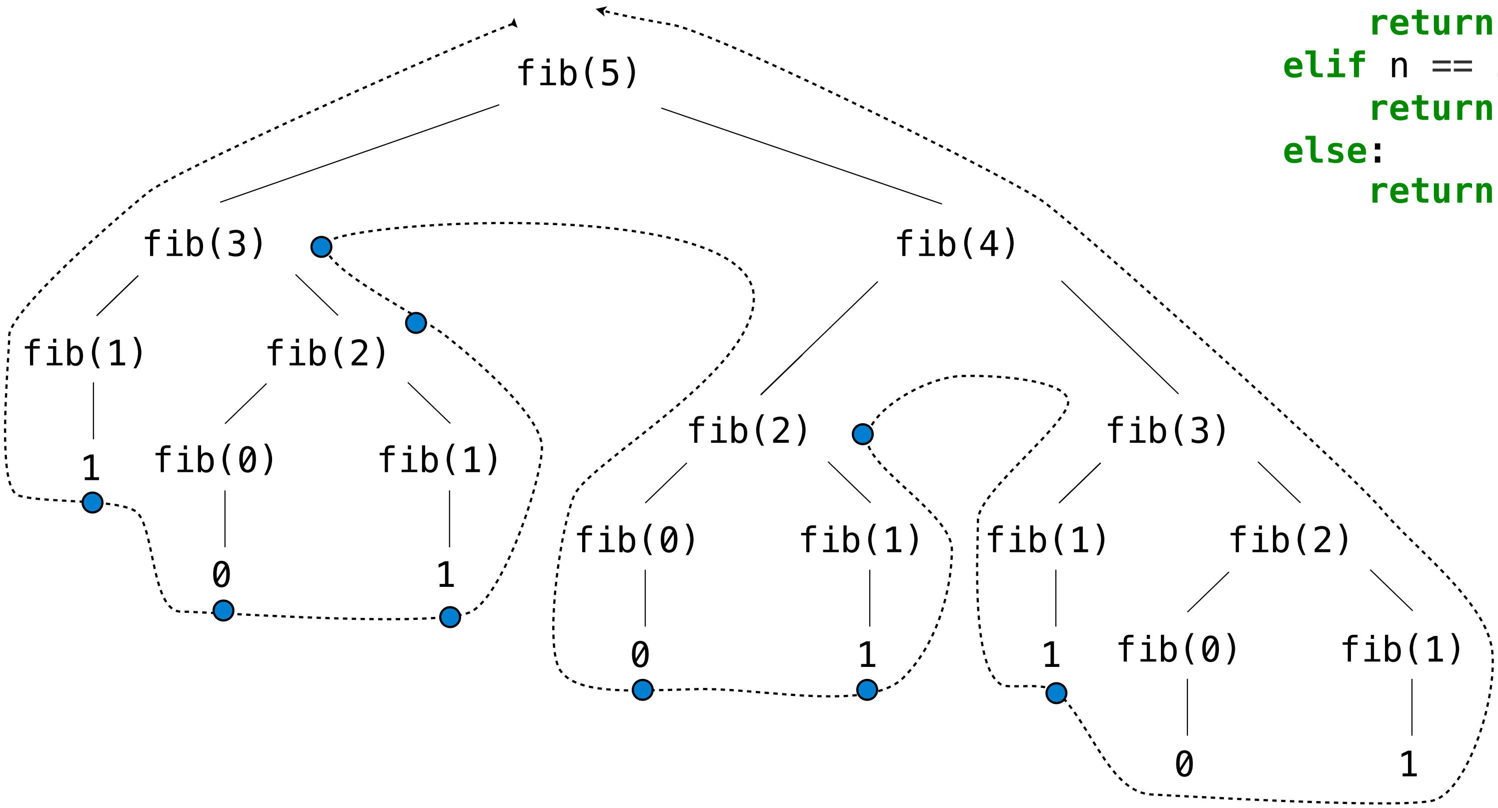


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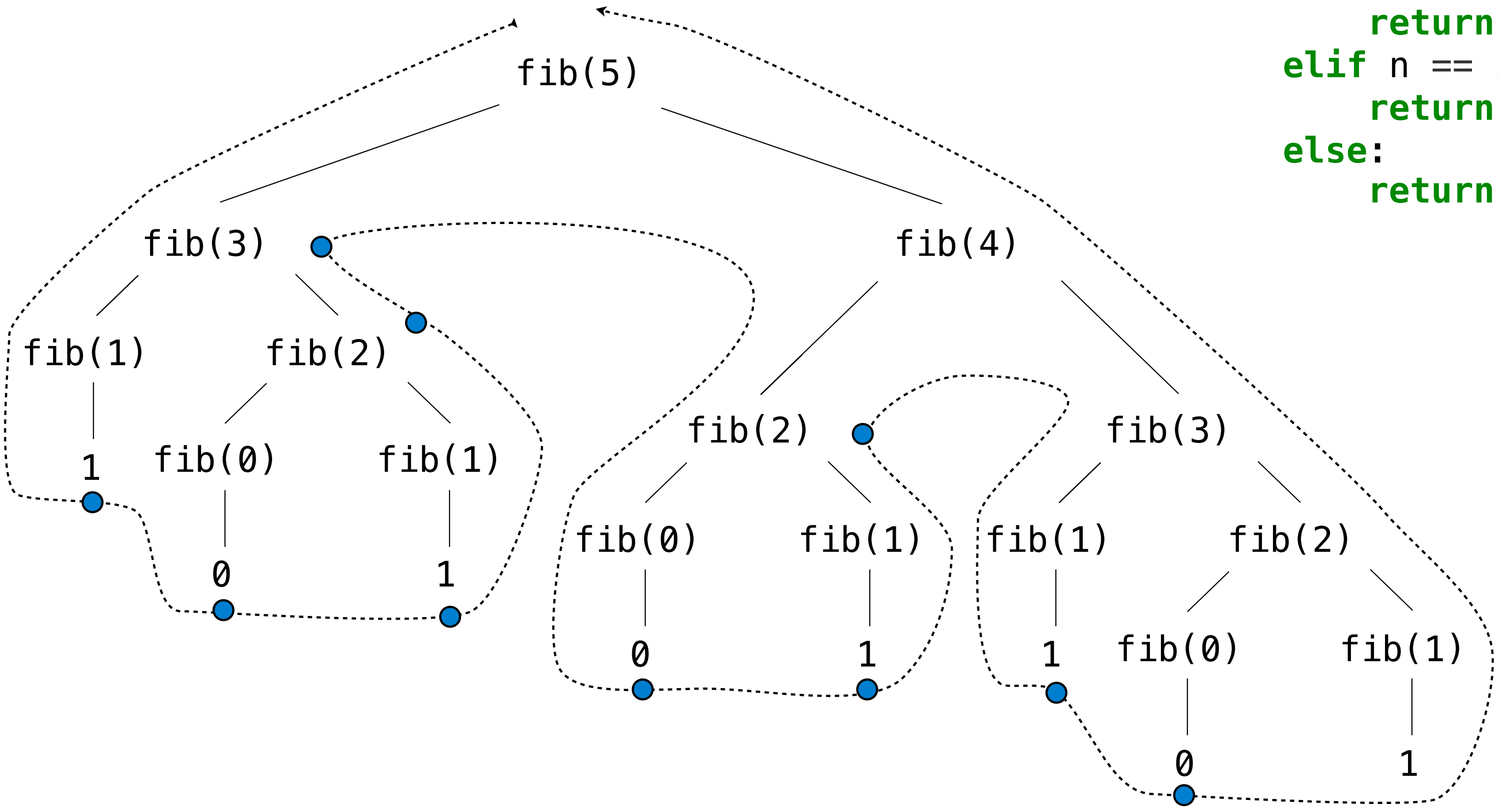

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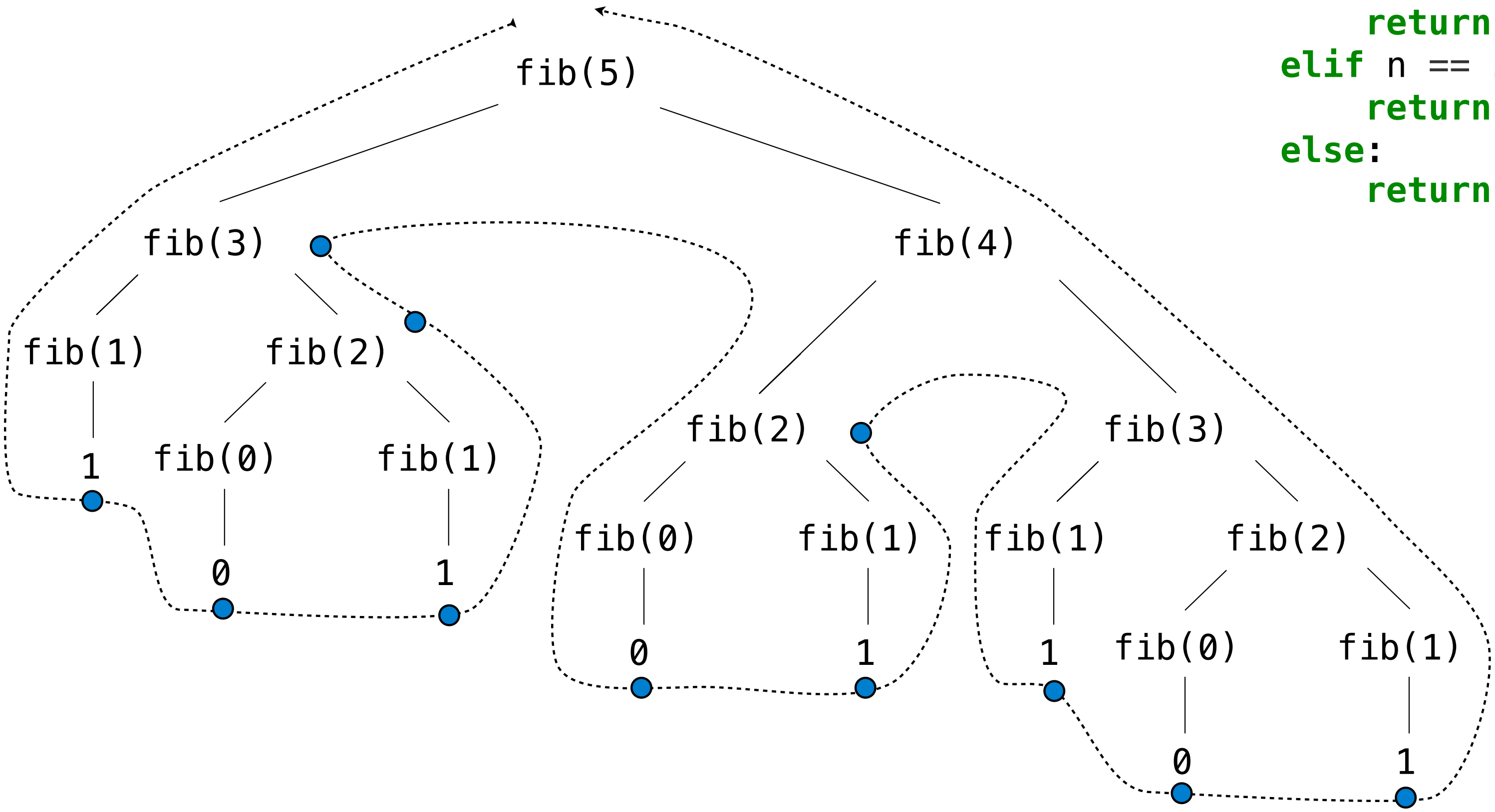

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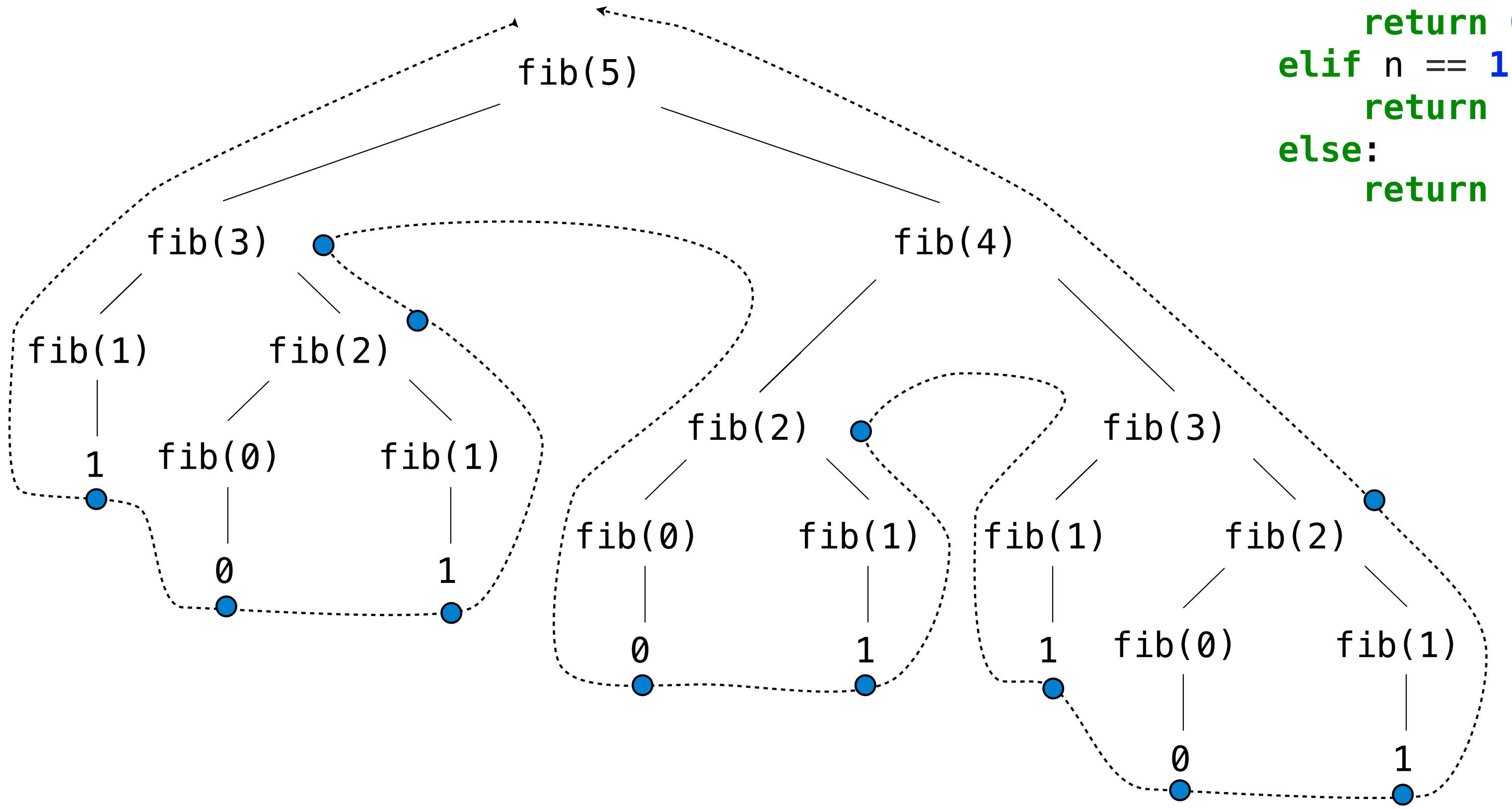


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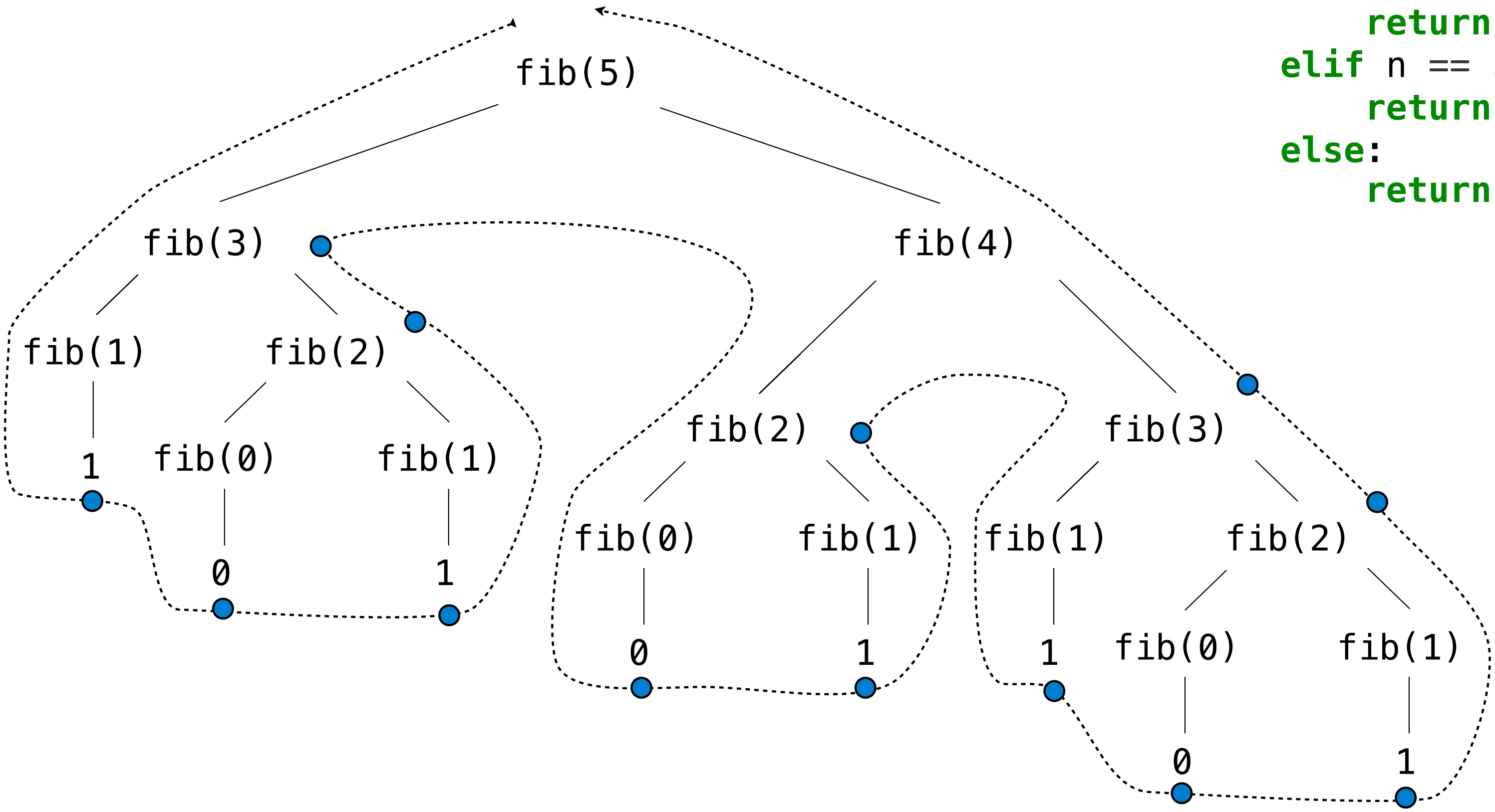

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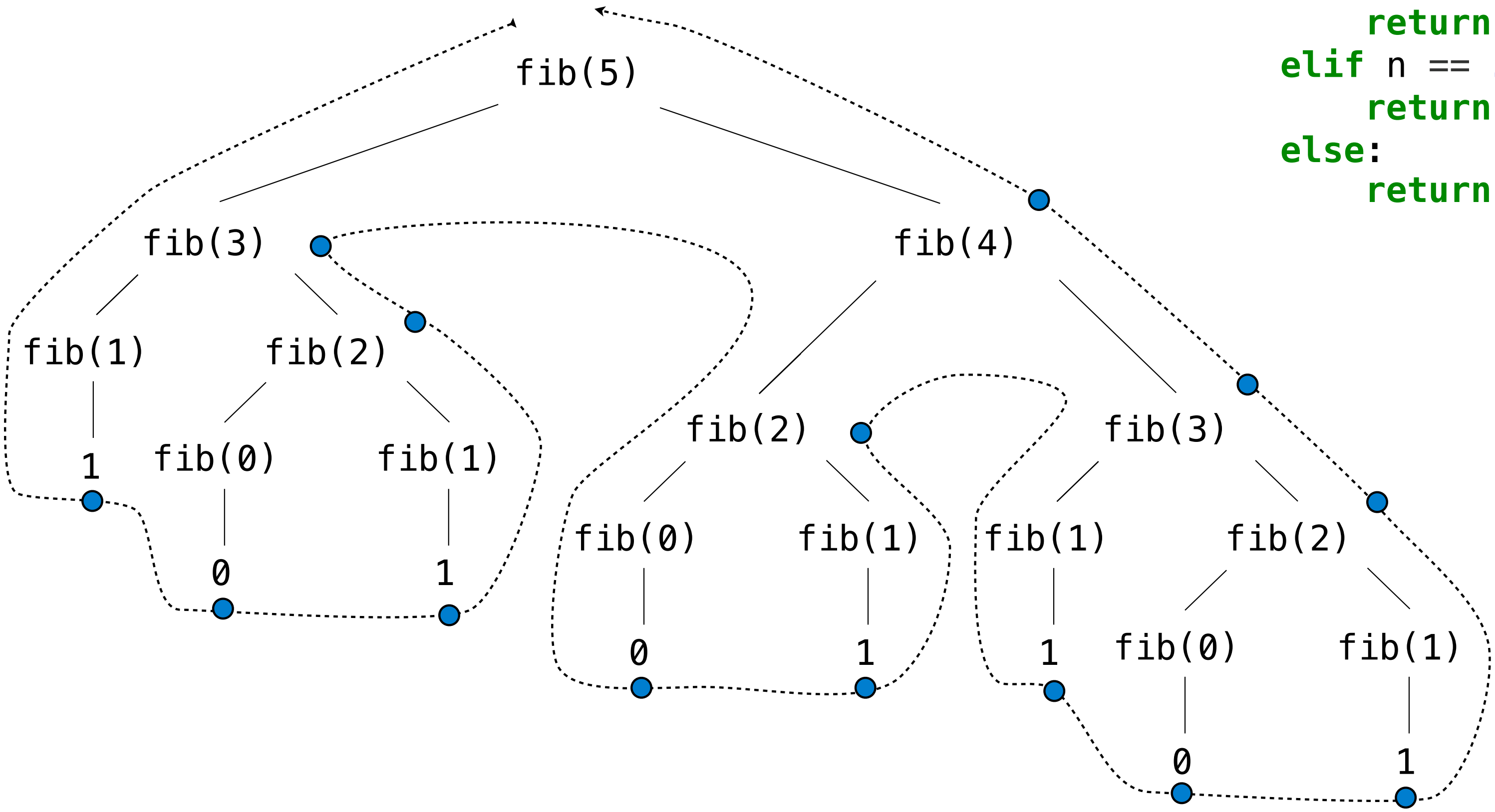

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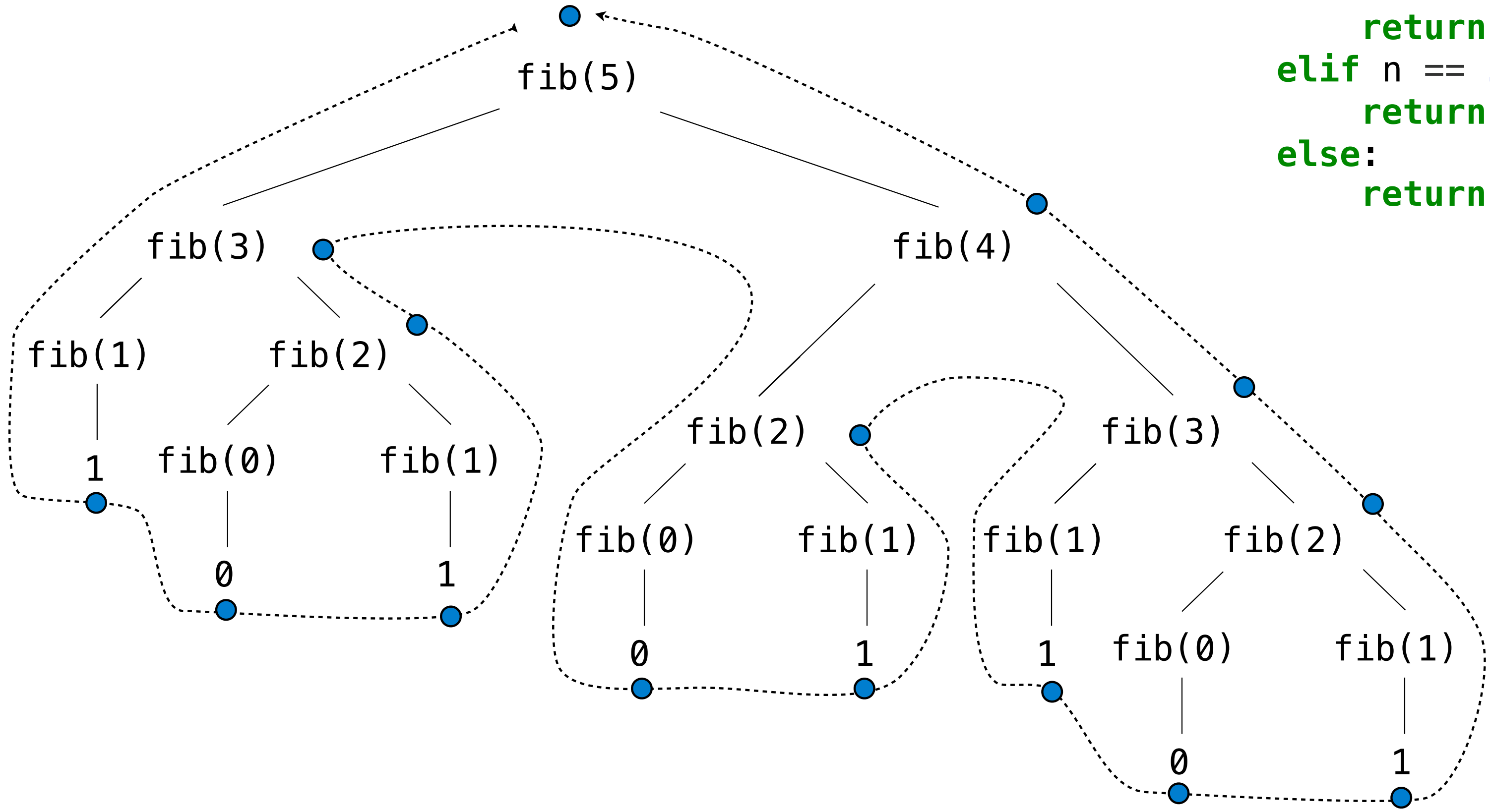

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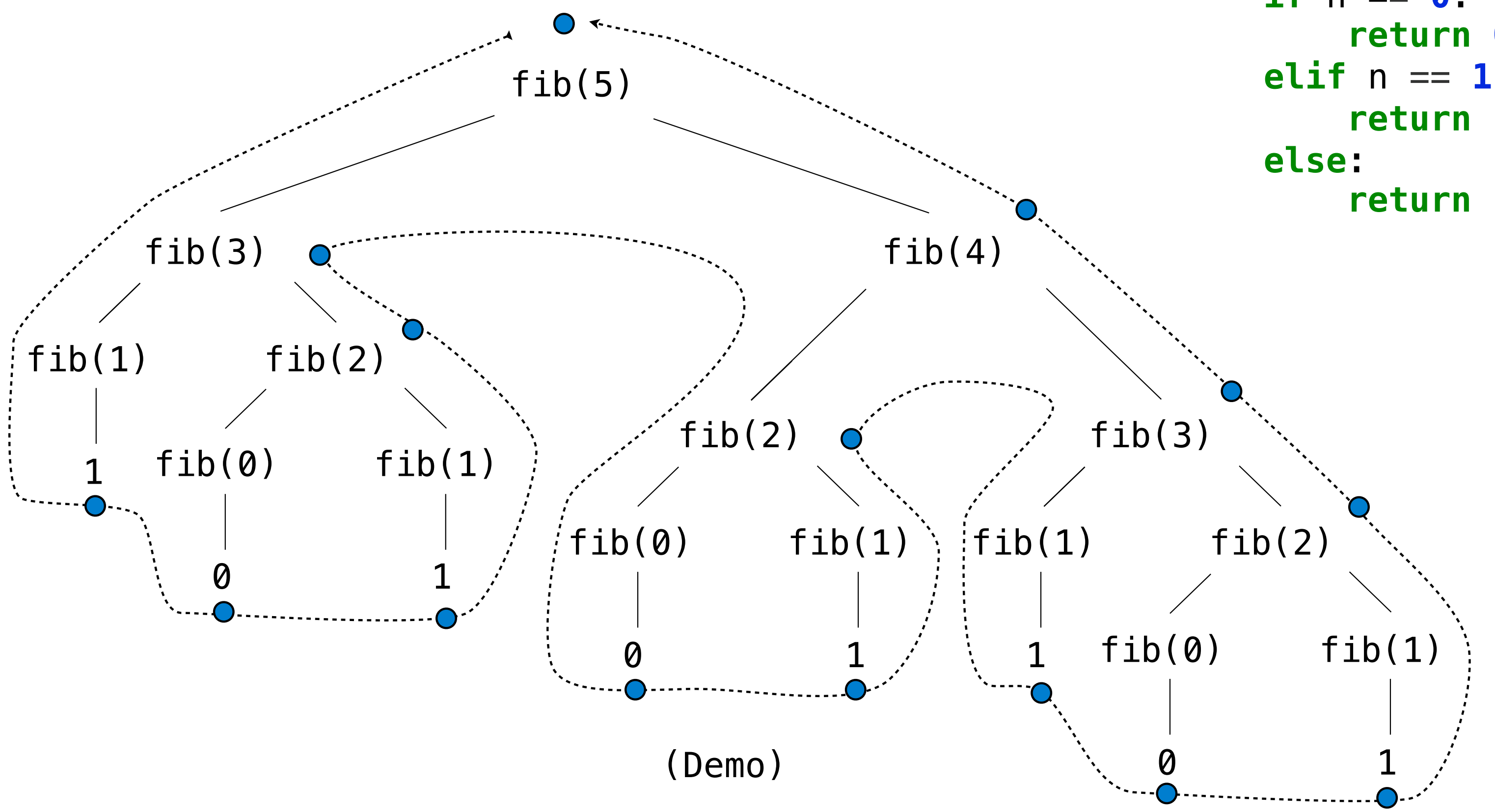

(Demo)

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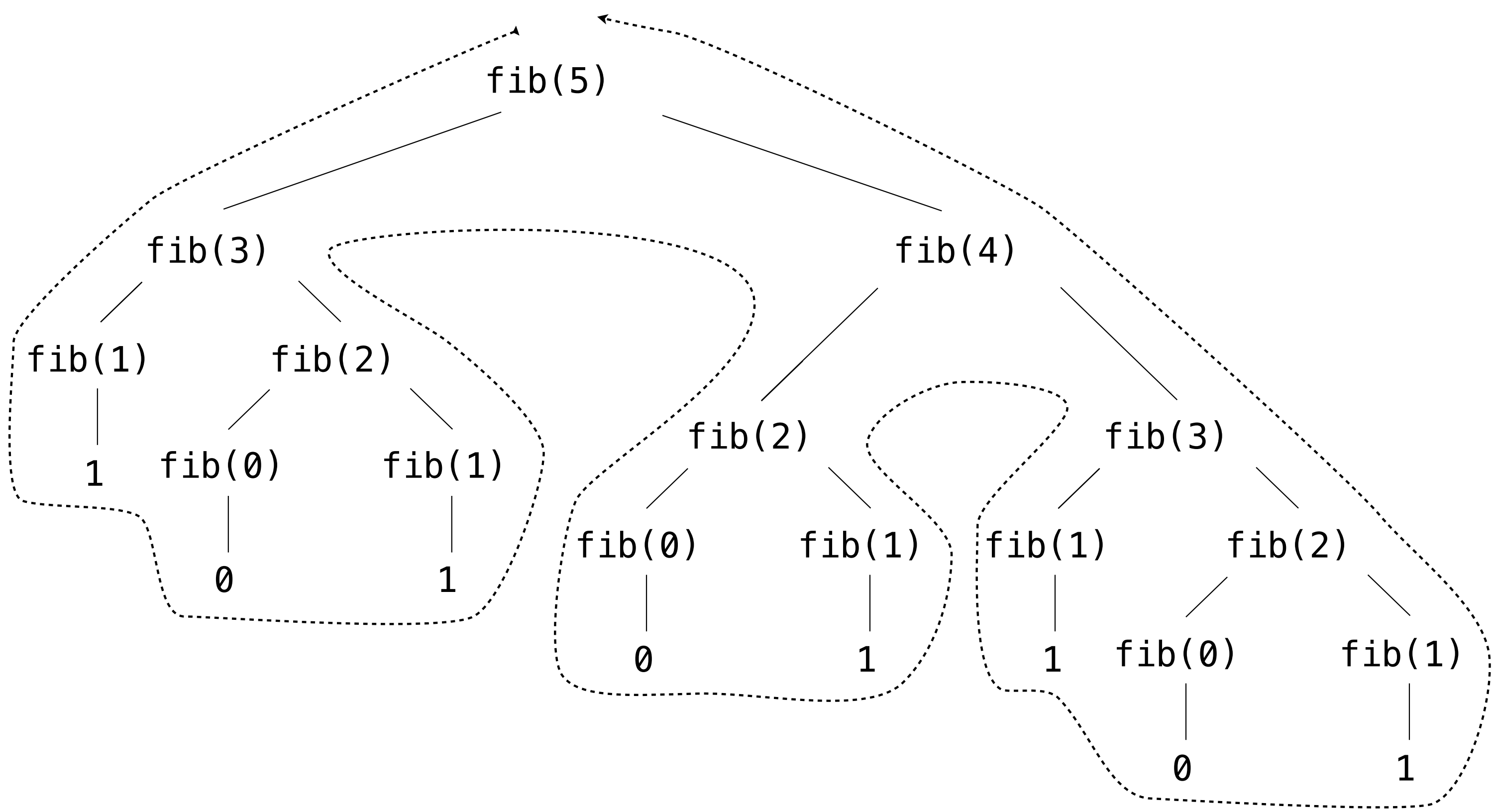

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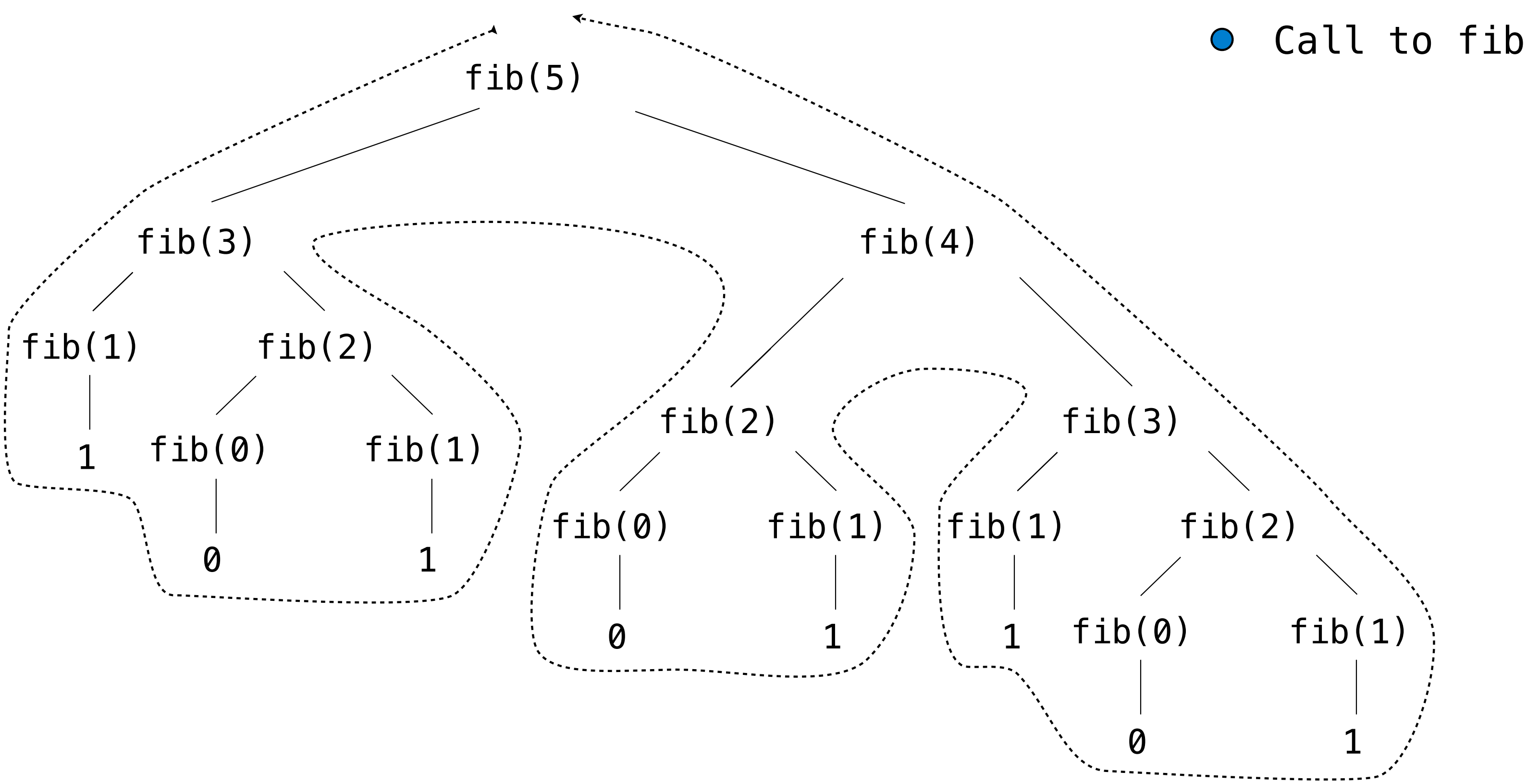

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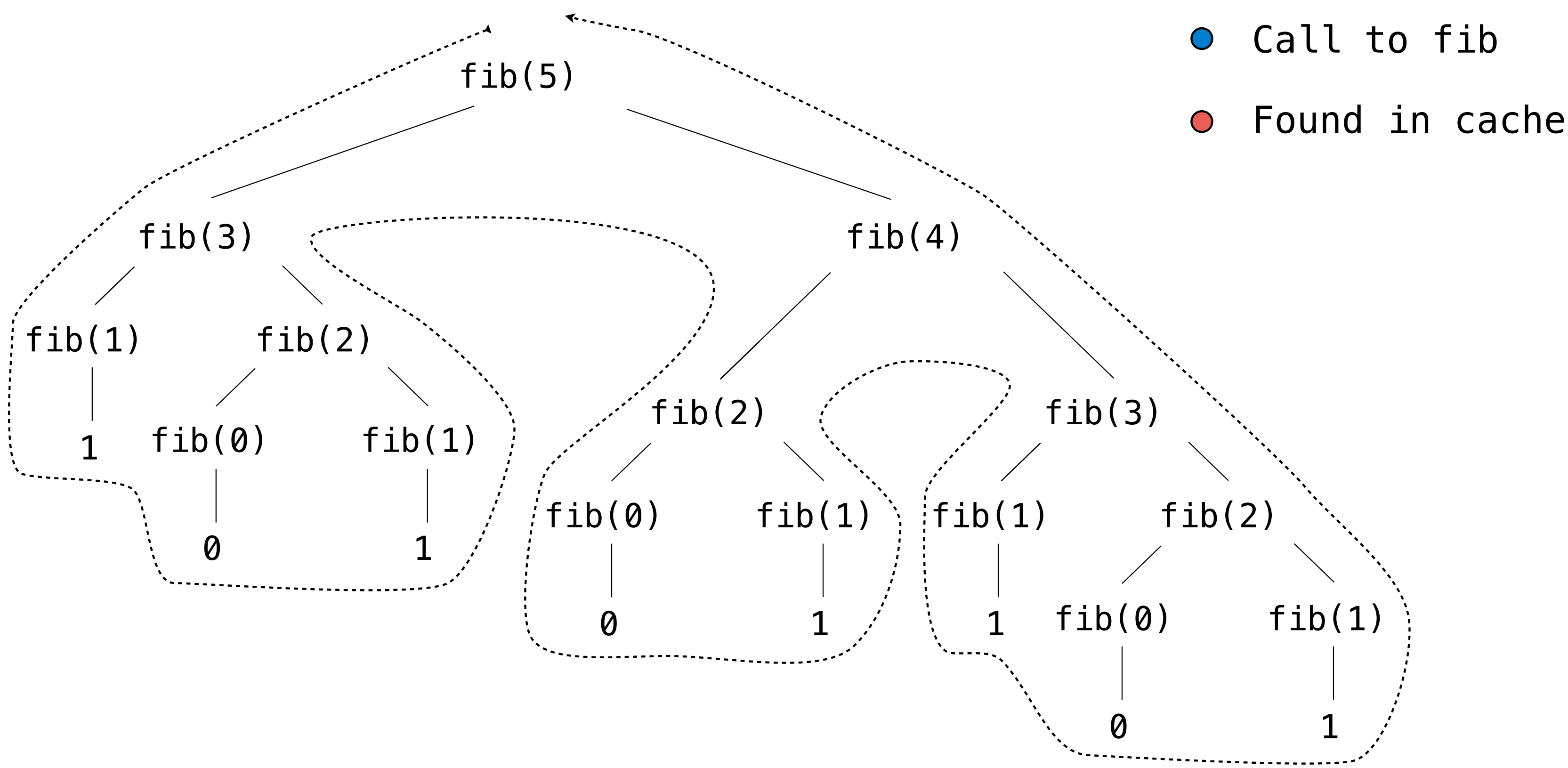

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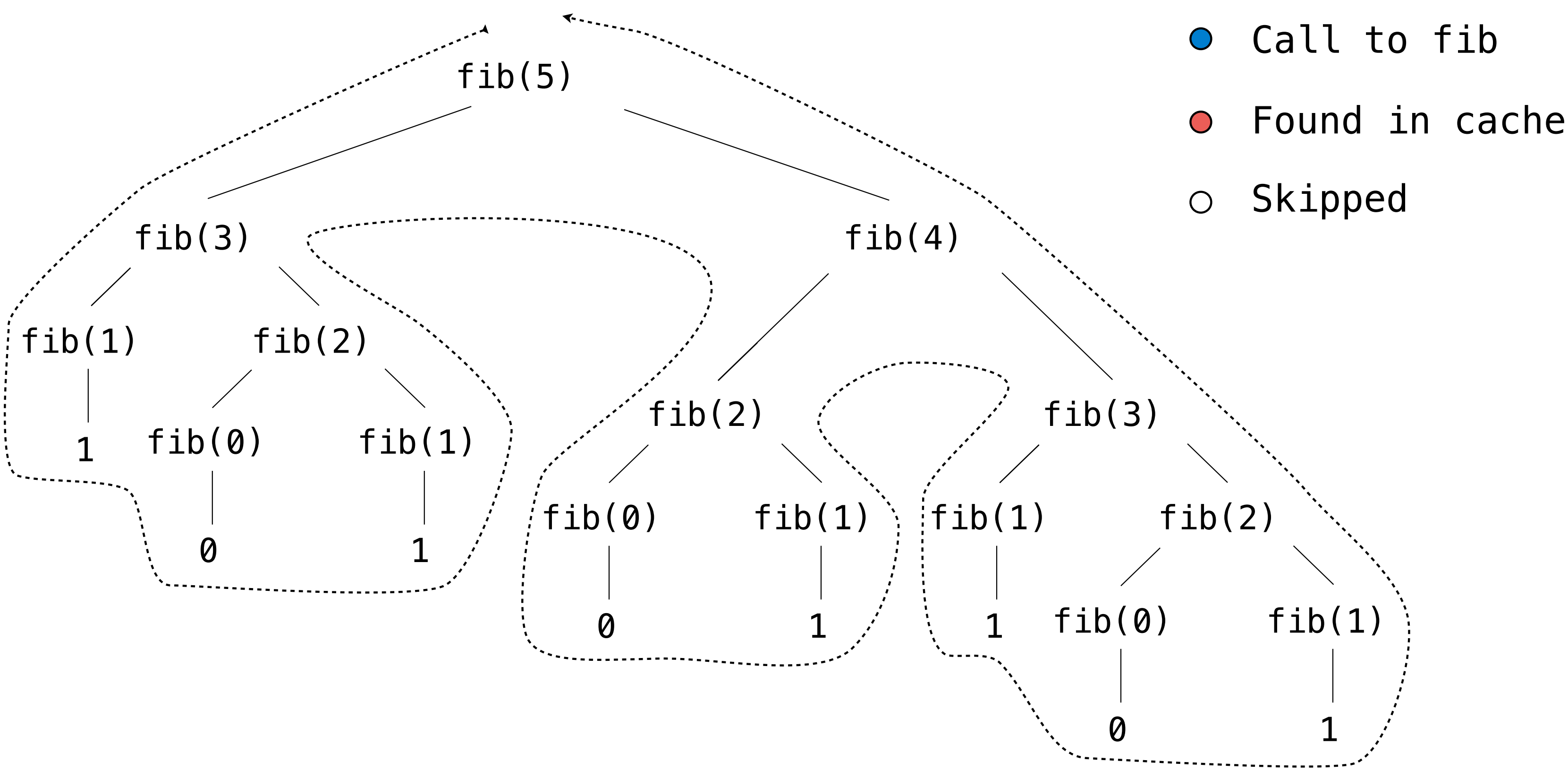

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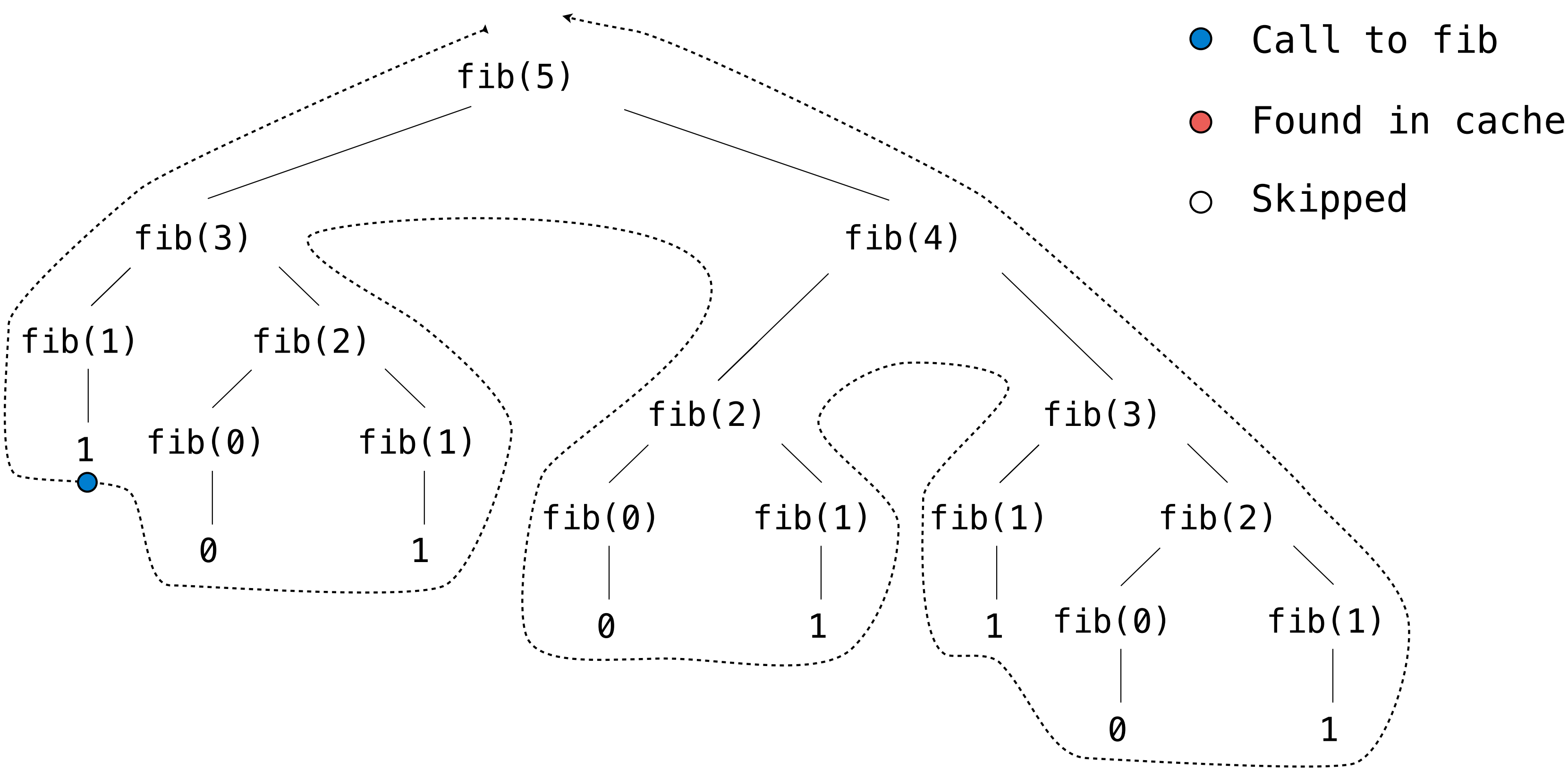

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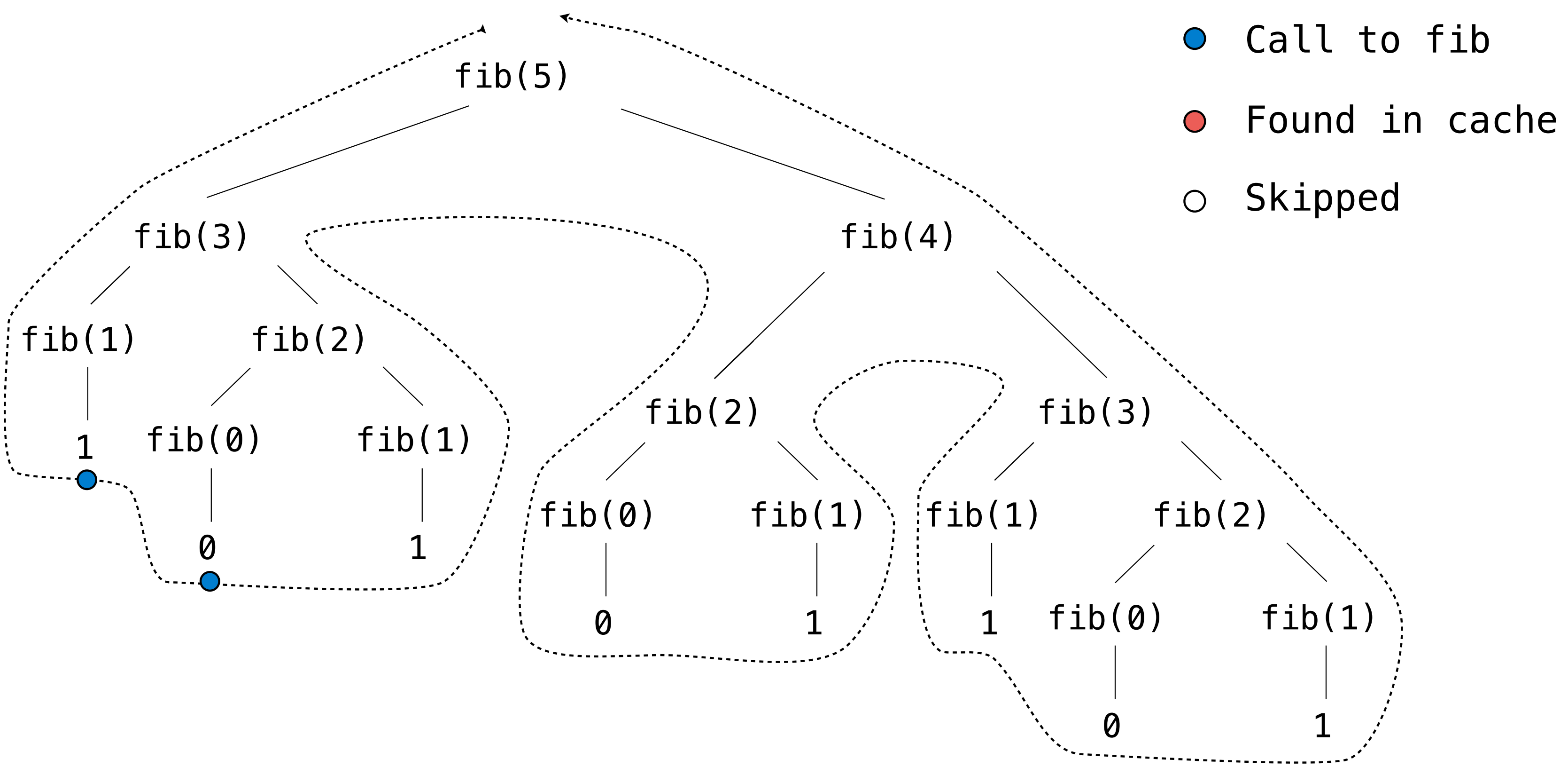

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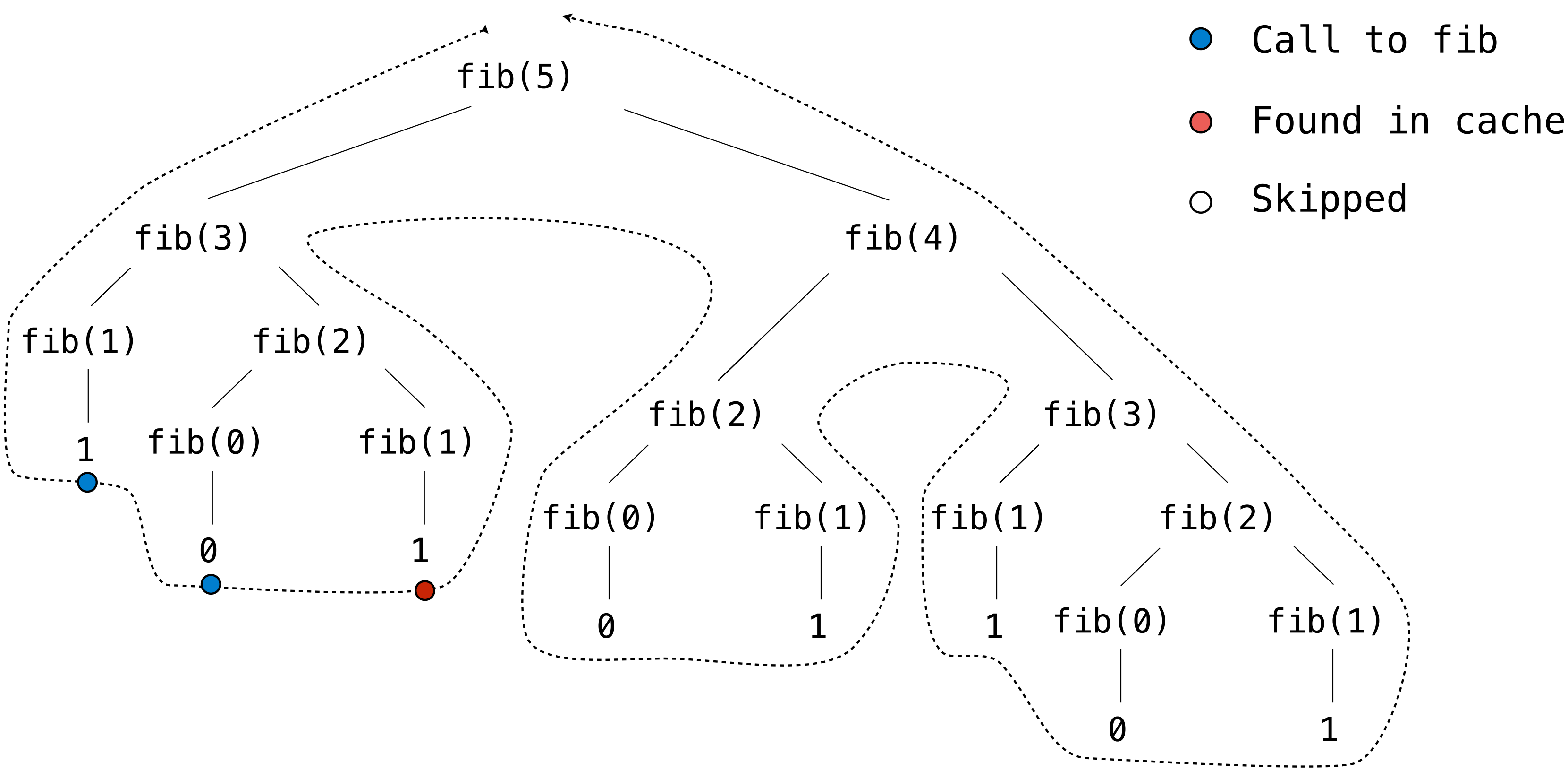

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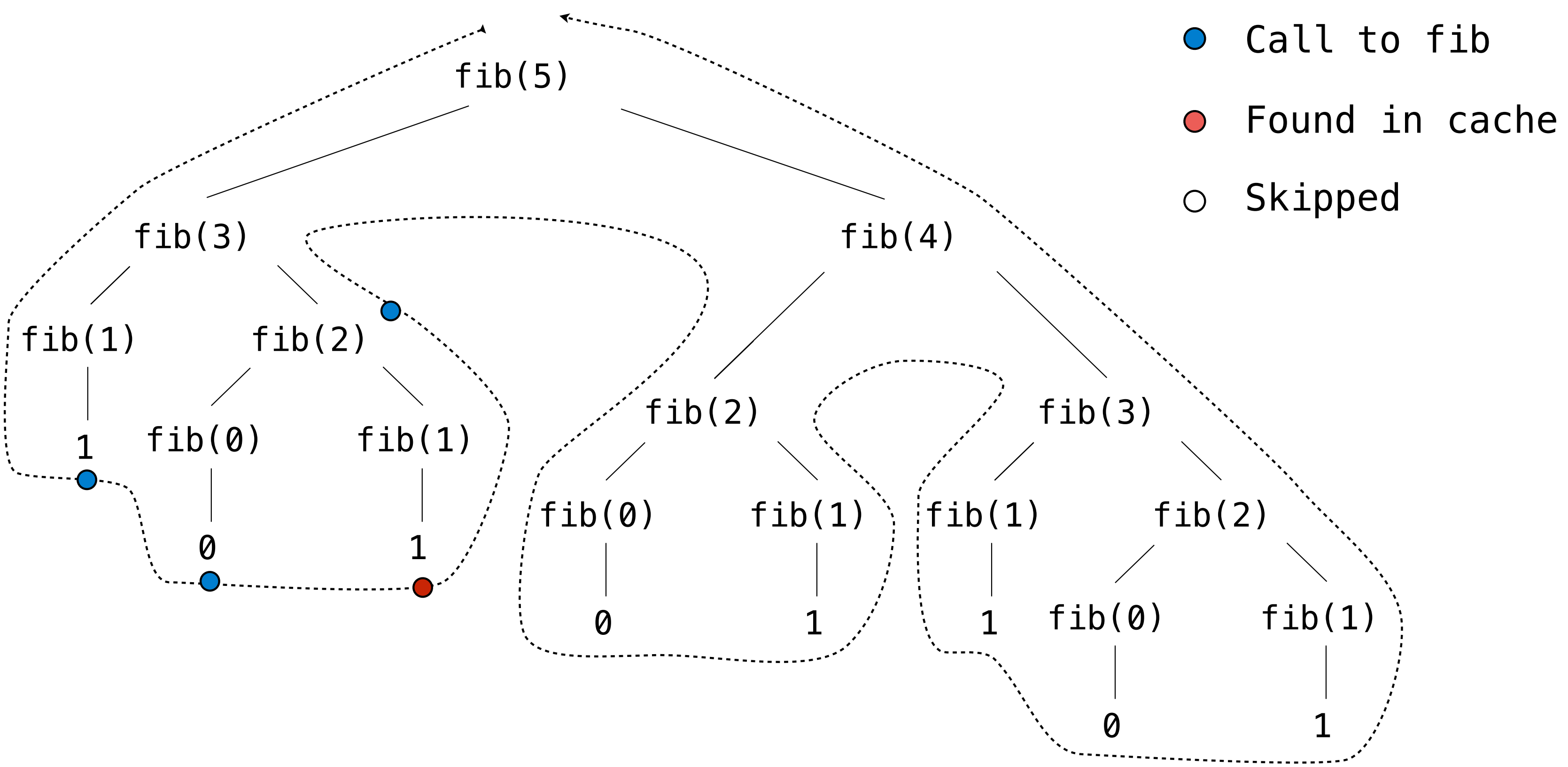

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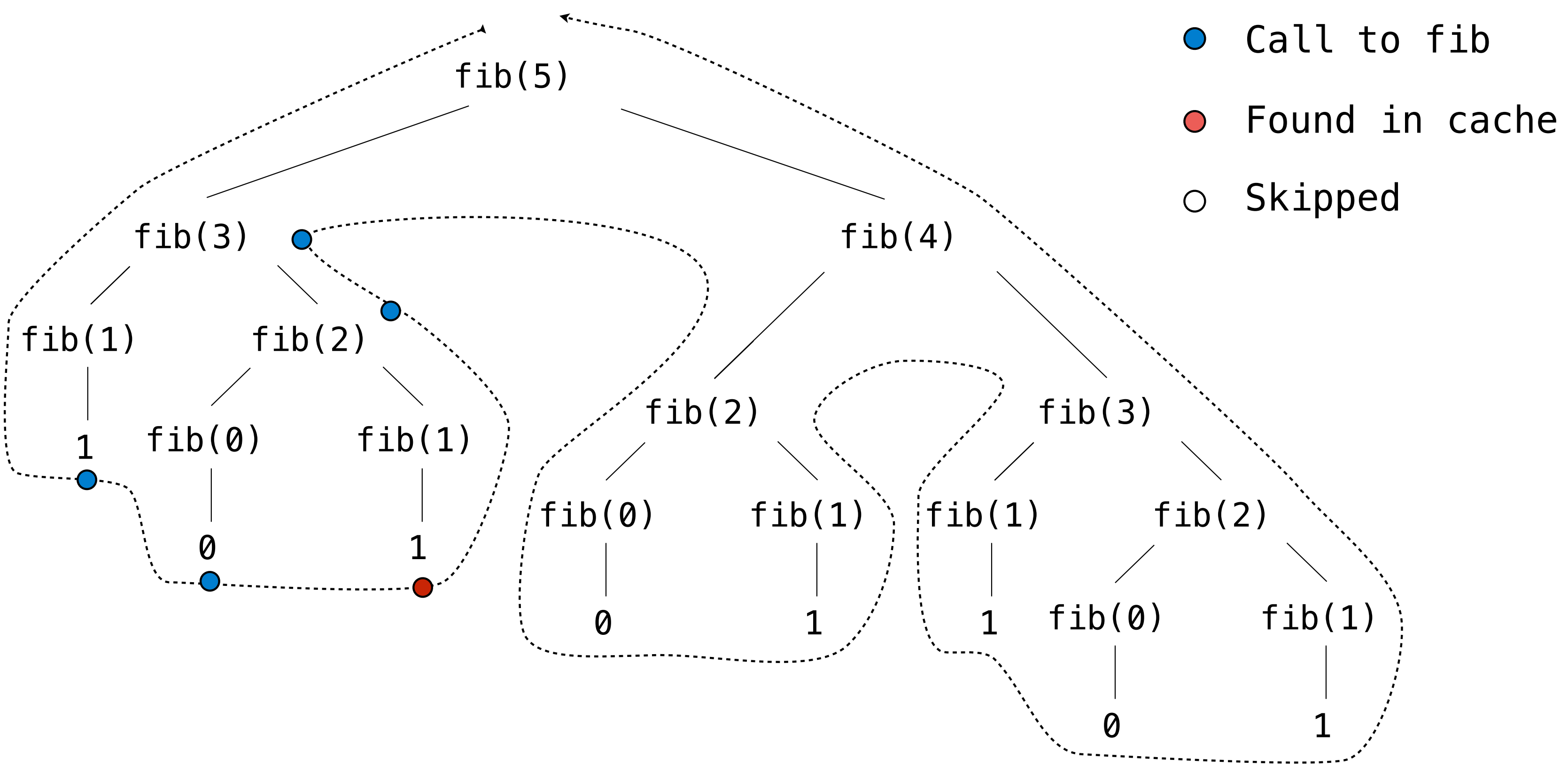

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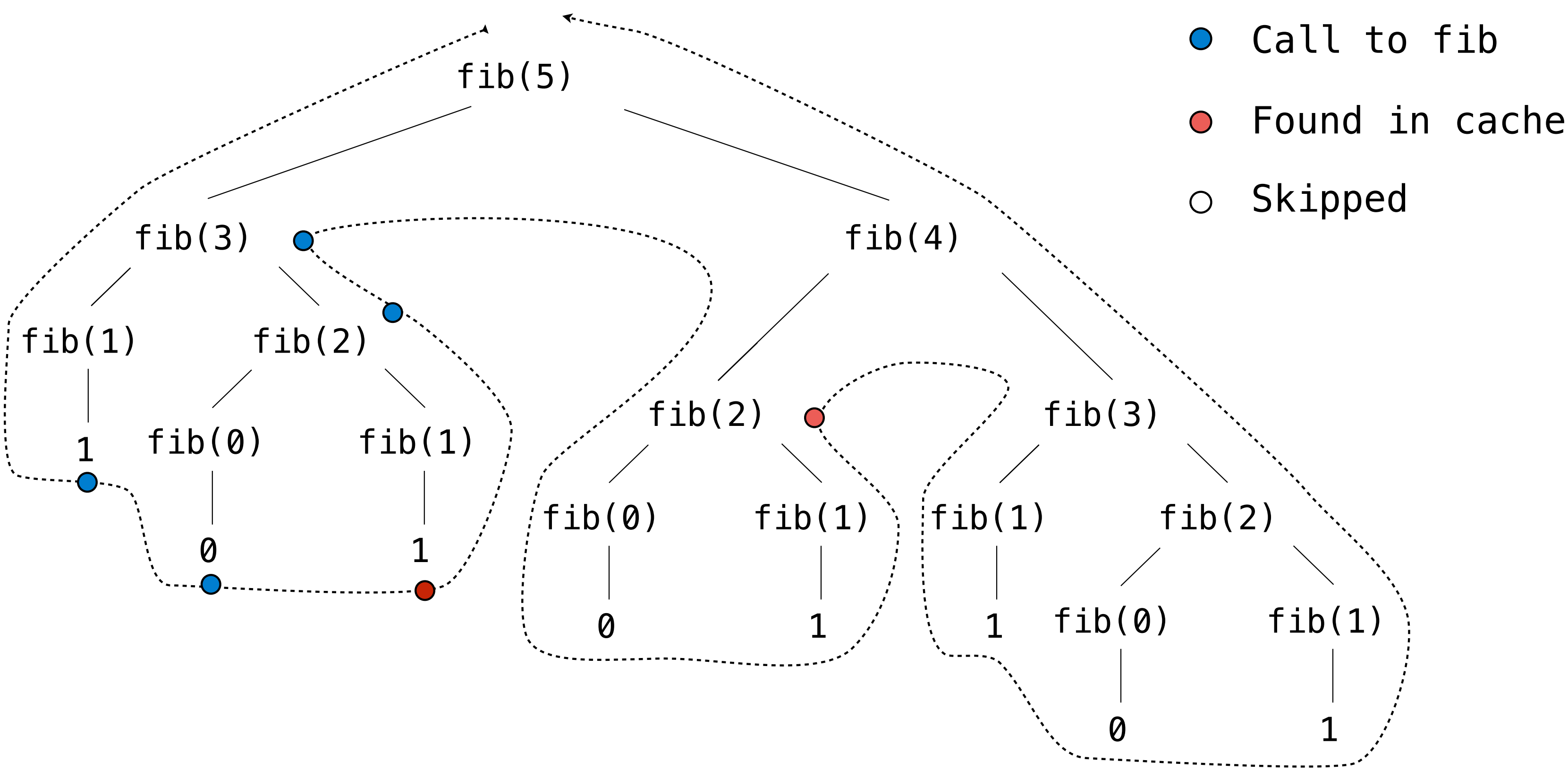

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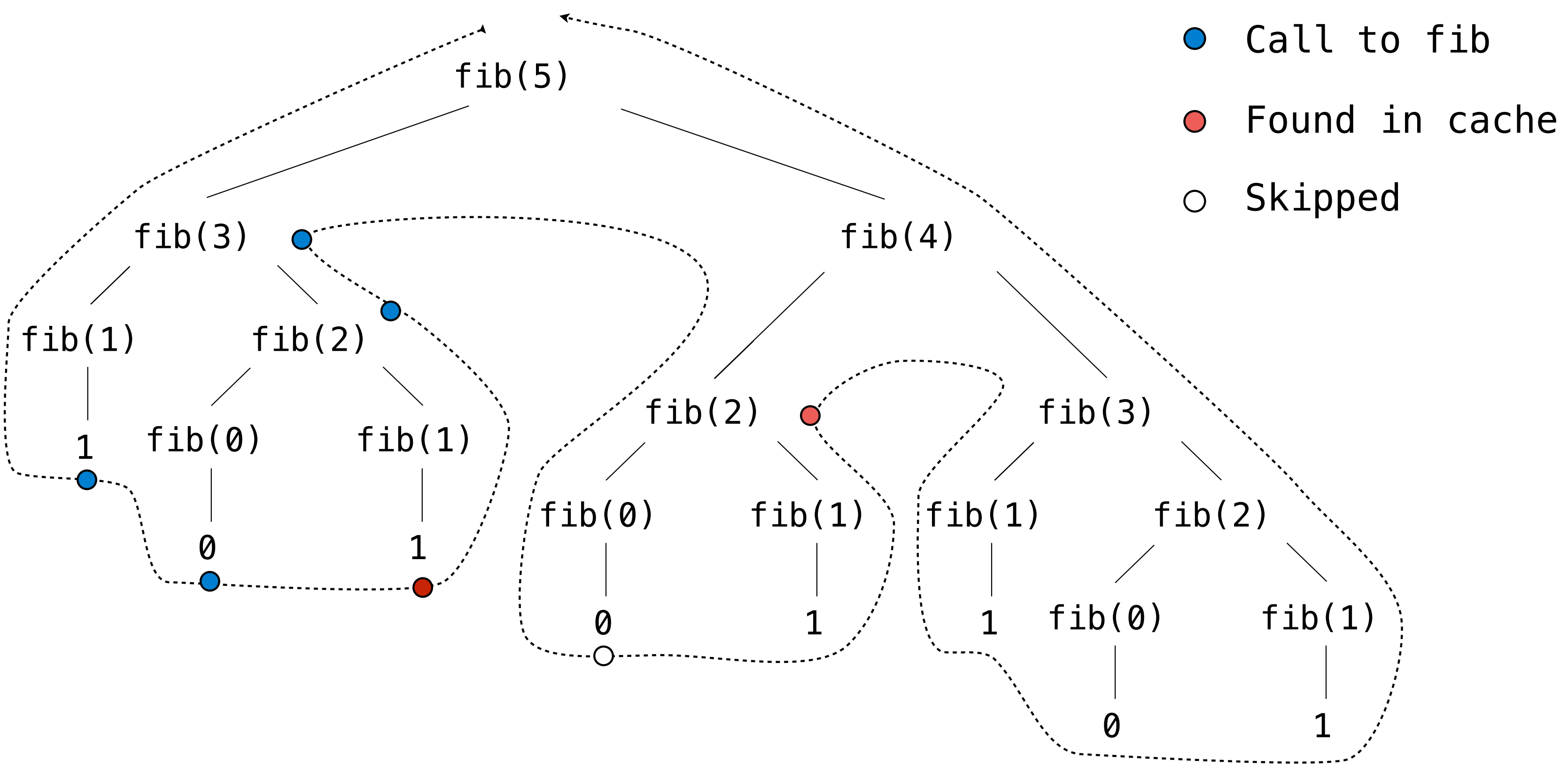

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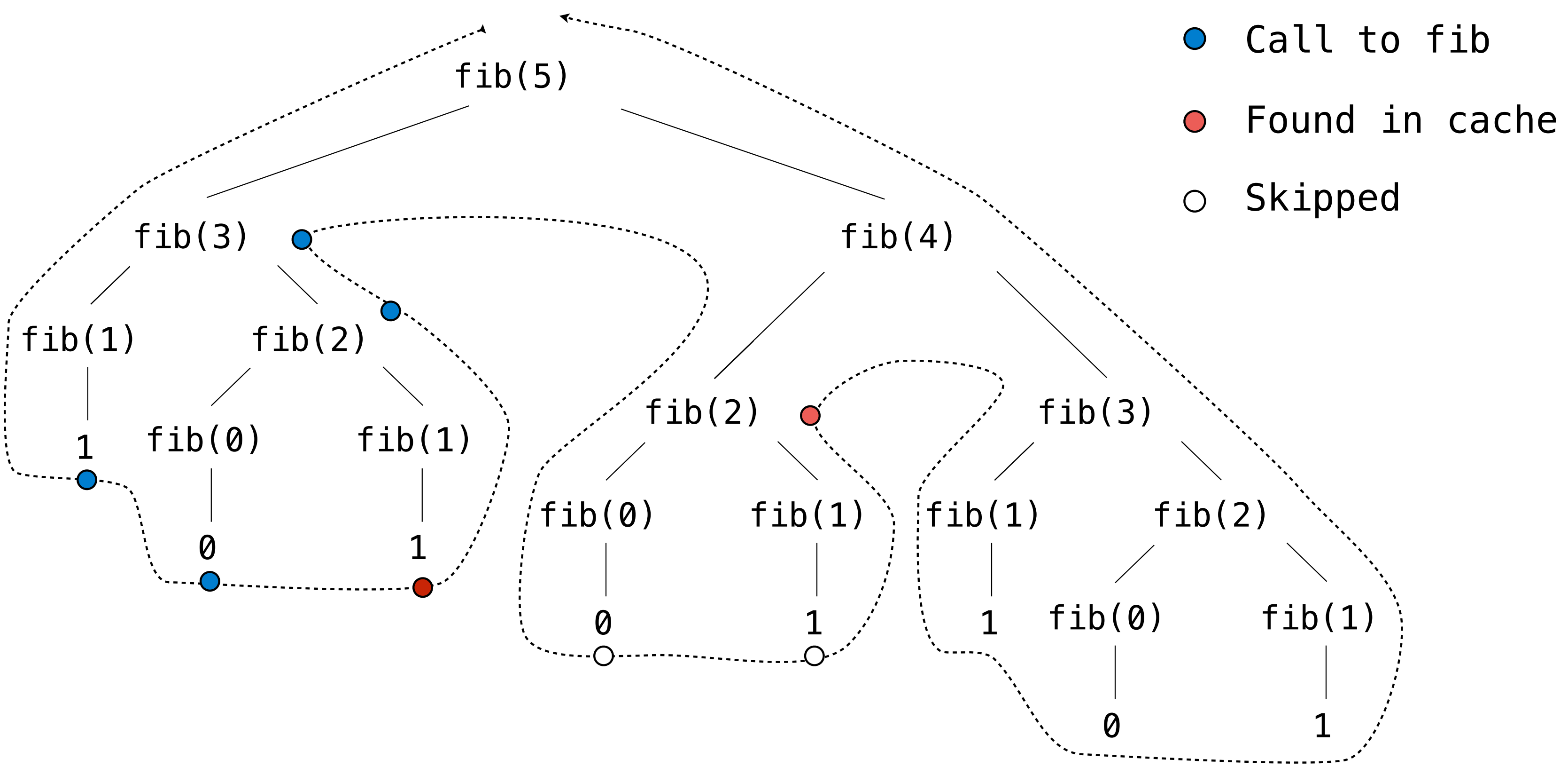

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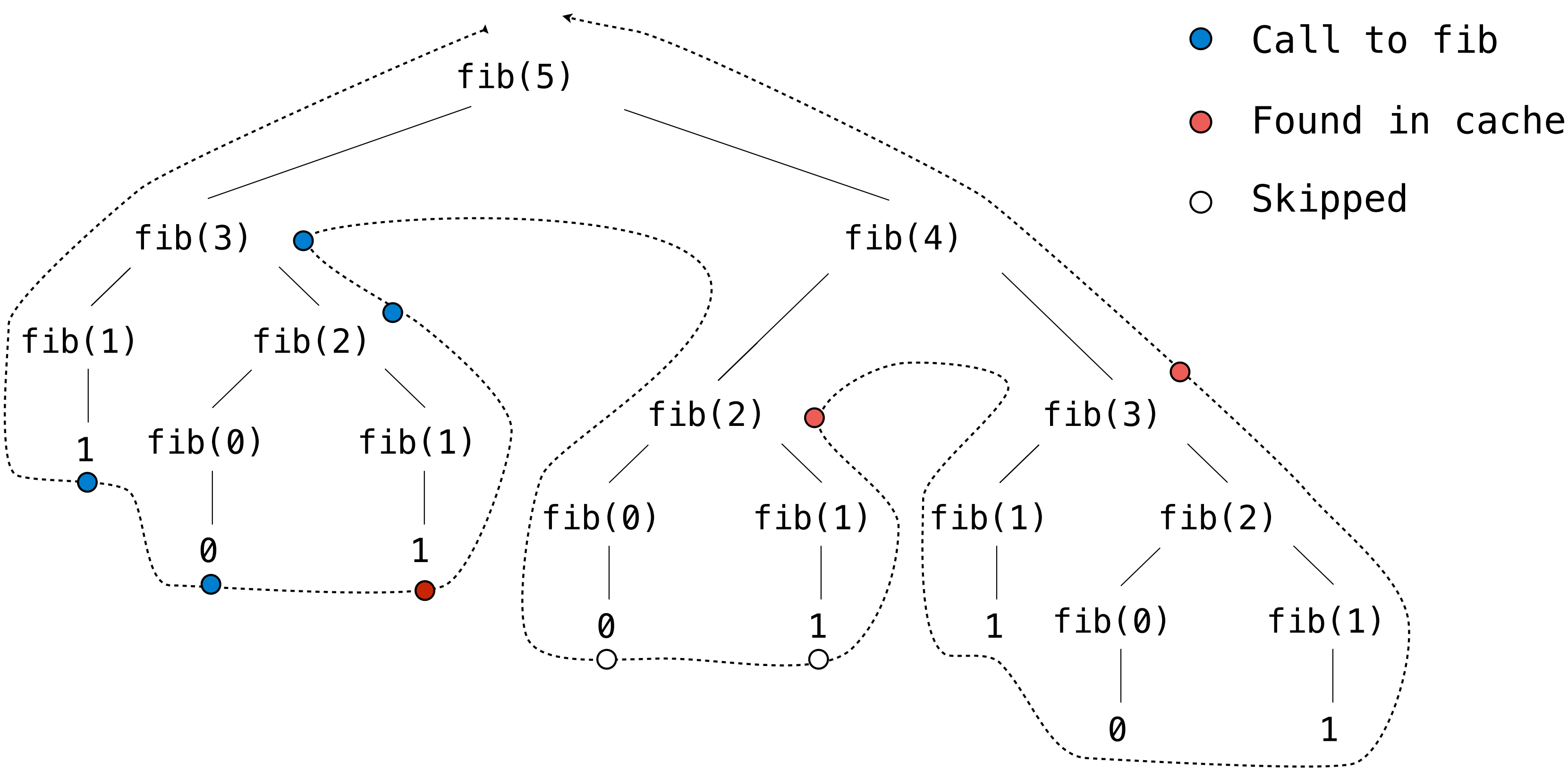

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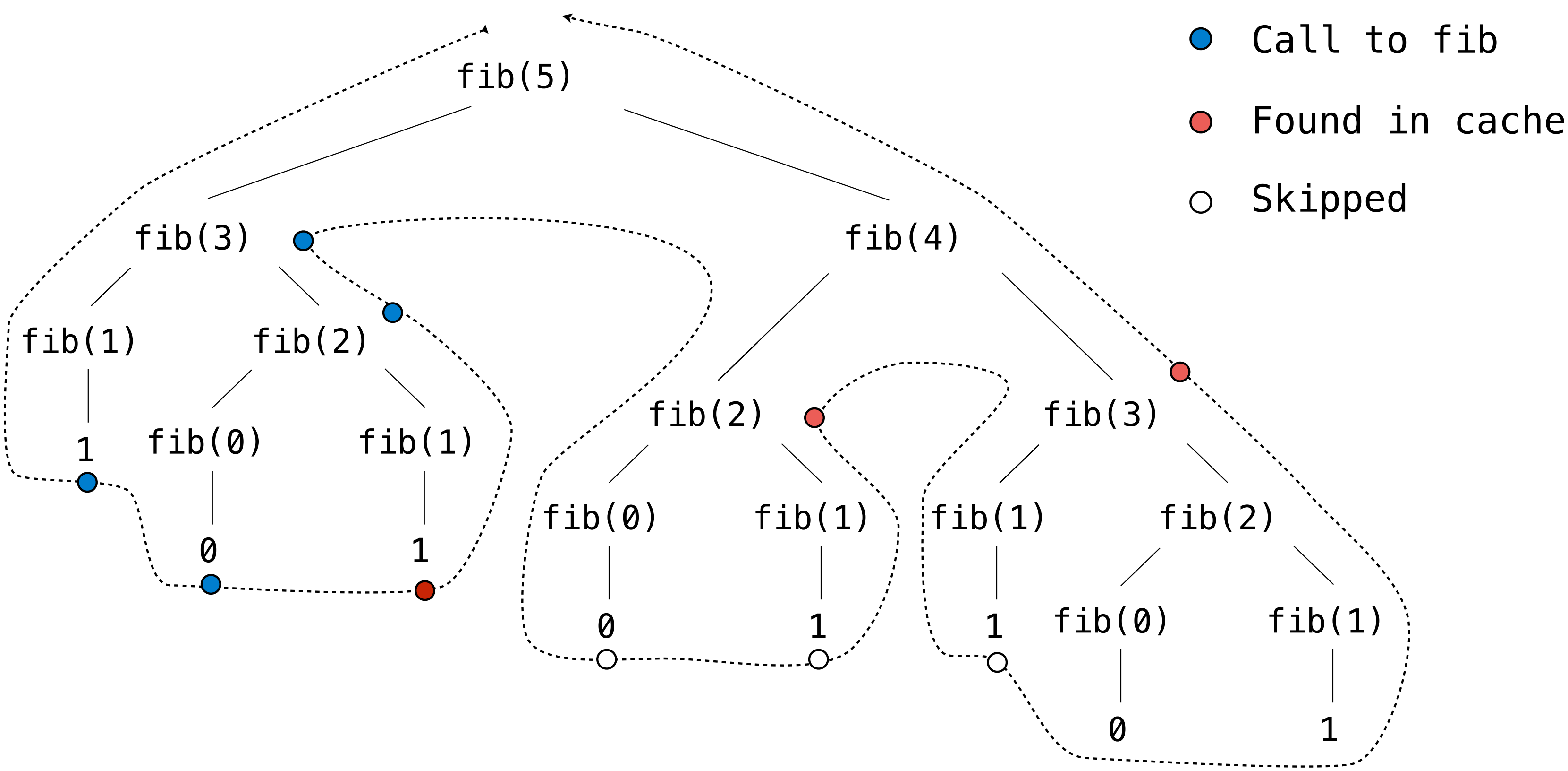

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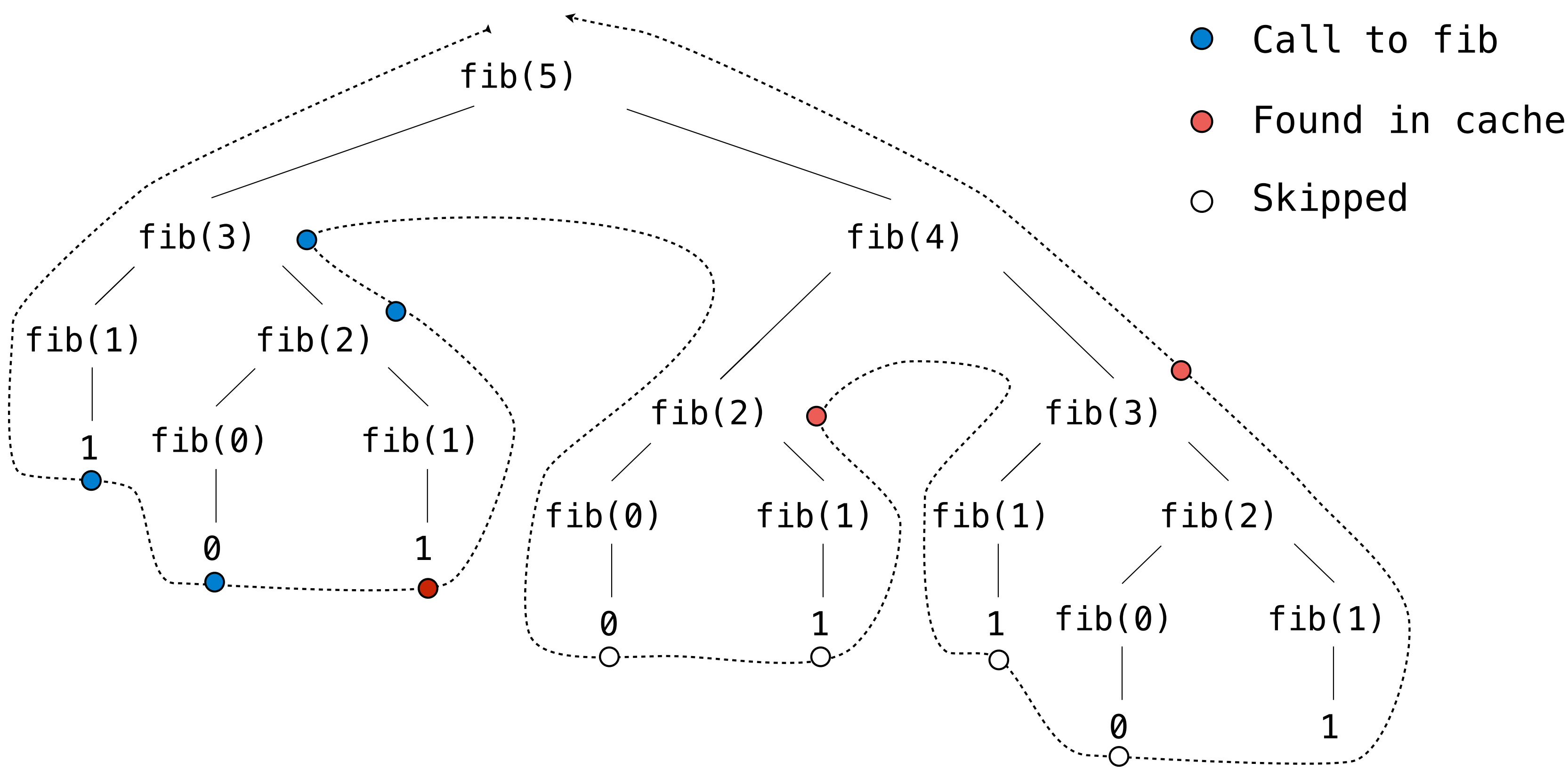

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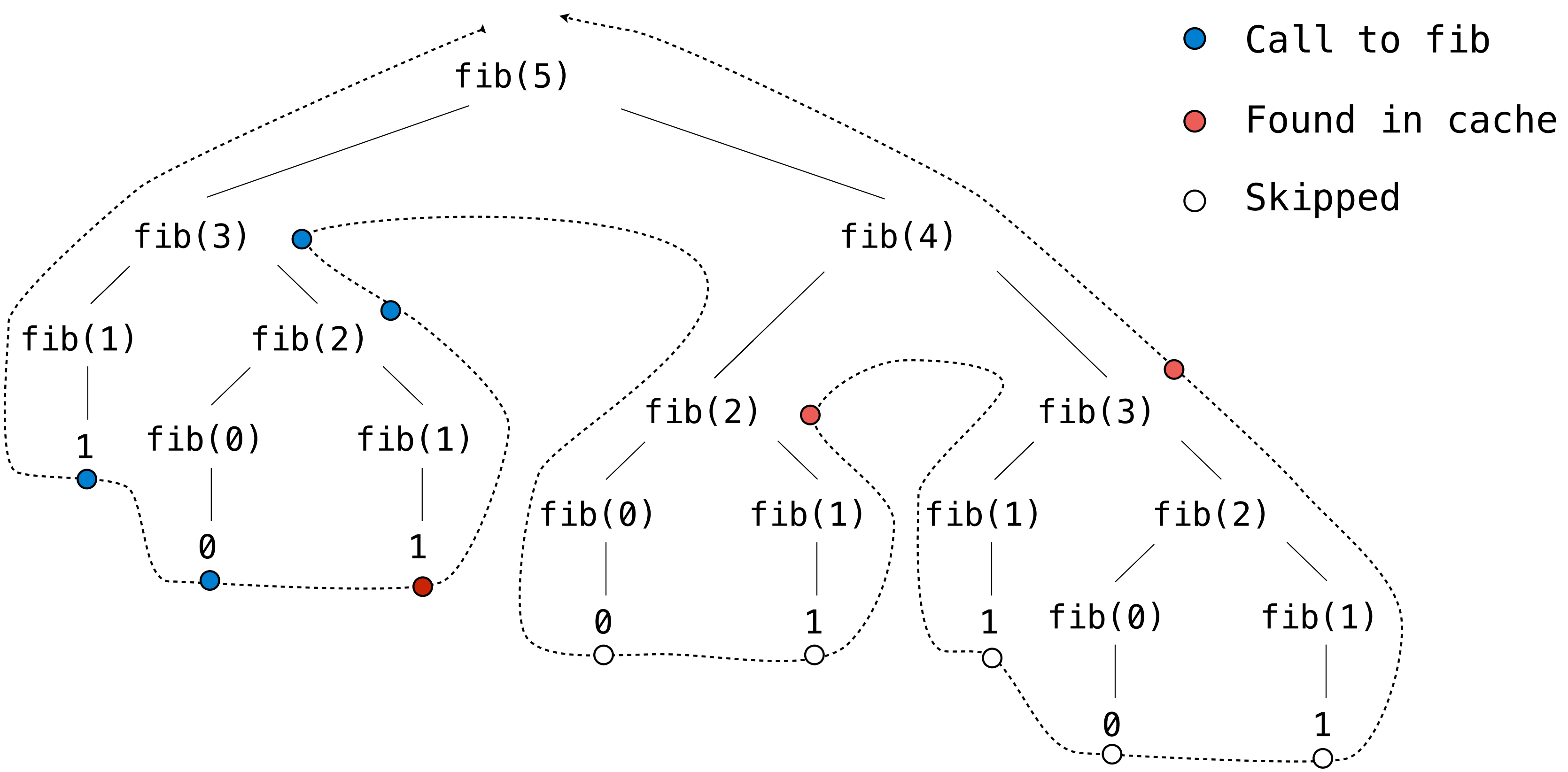

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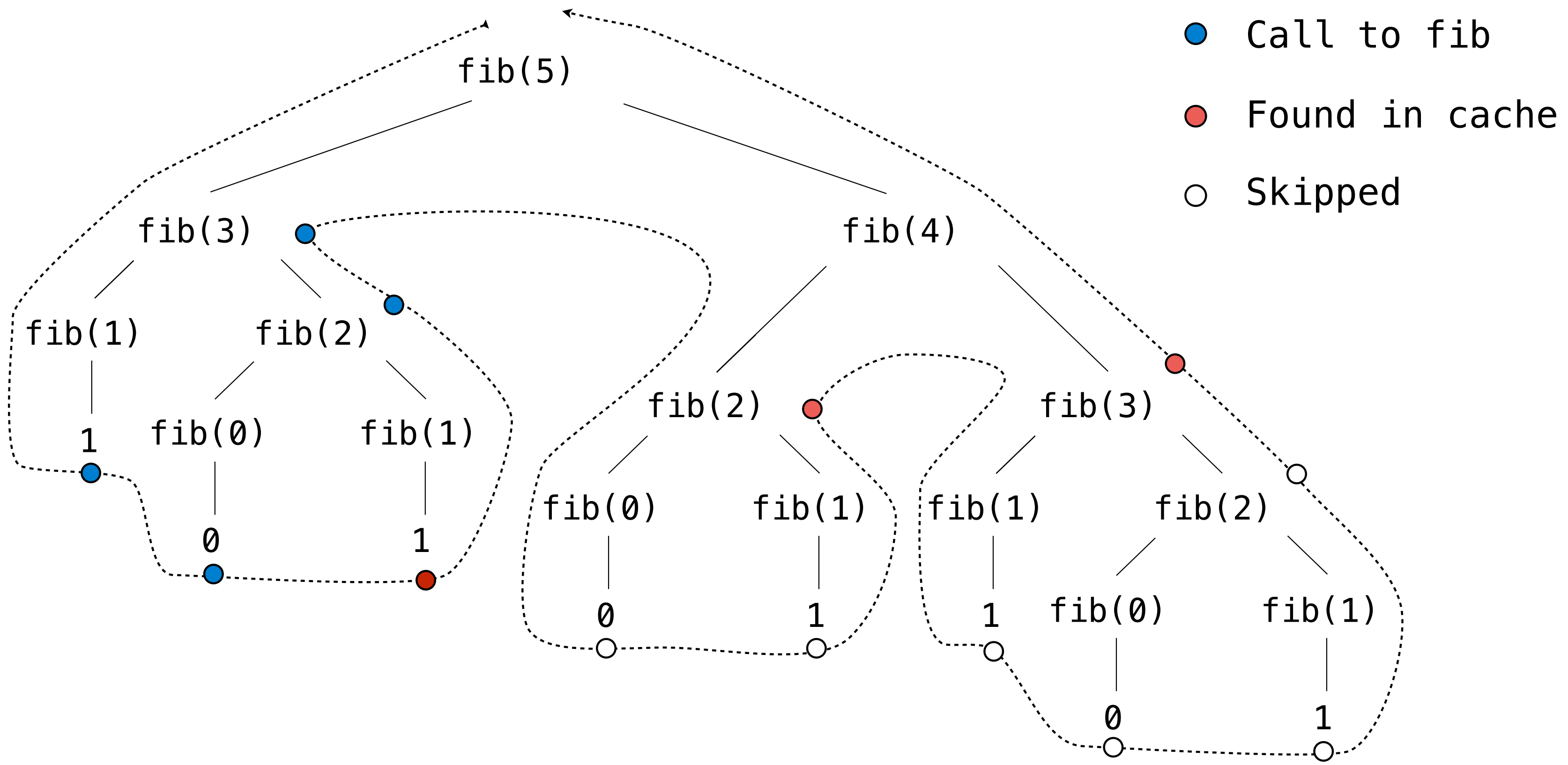

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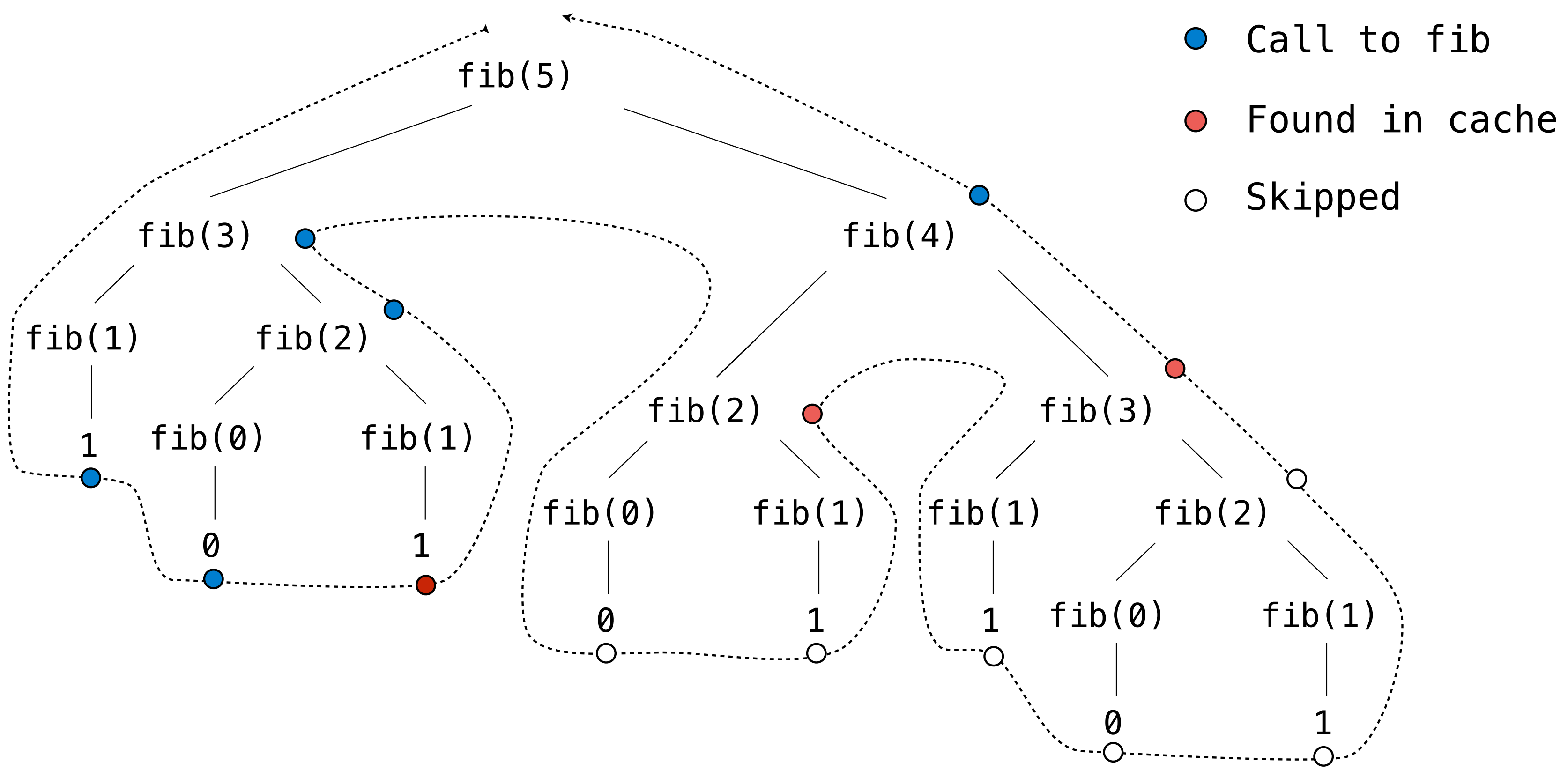

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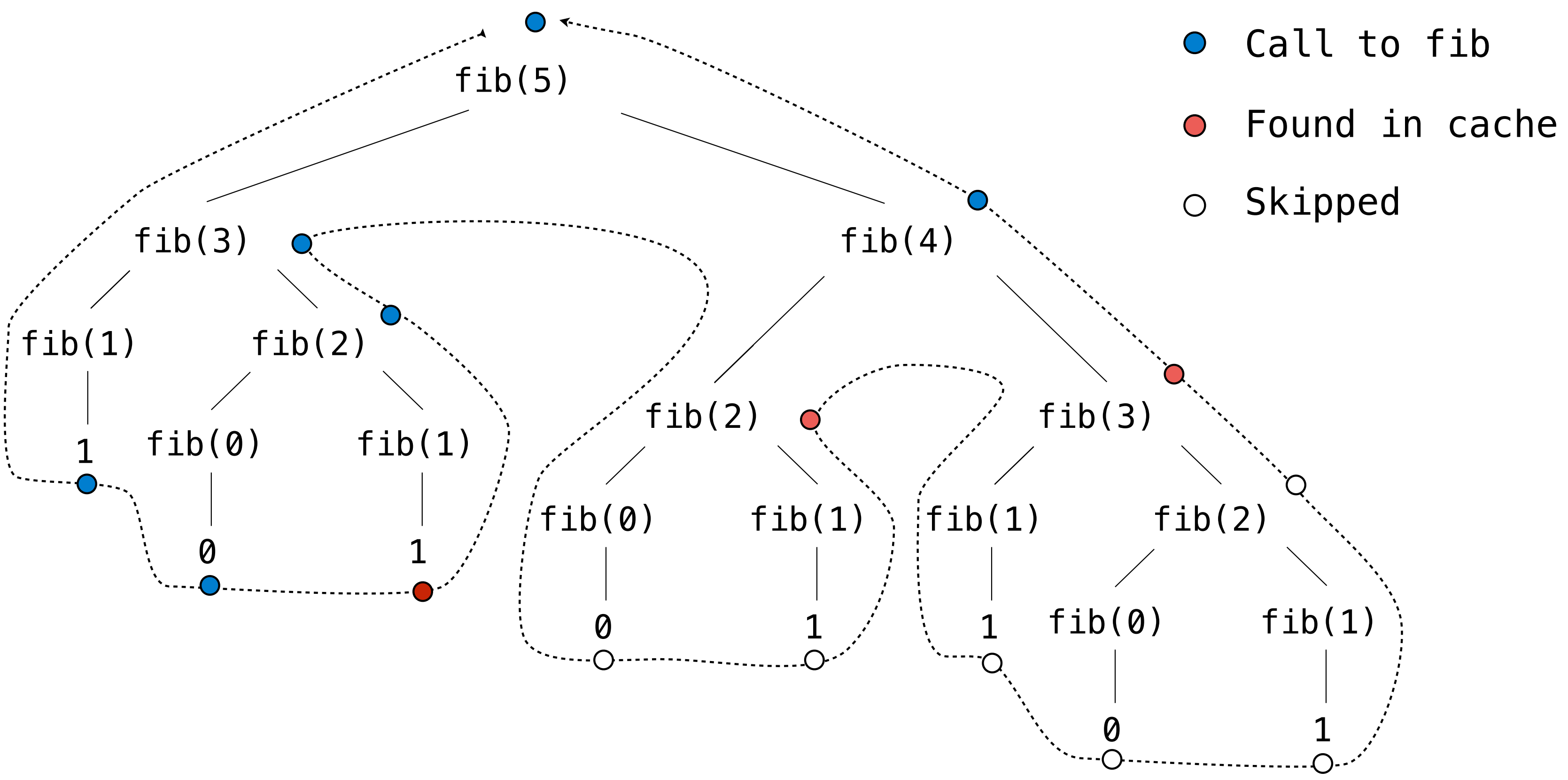

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def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

def square(x):
    return x * x
```

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        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

def square(x):
    return x * x
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{n-1} & \text{if } n \text{ is odd} \end{cases}$$

(Demo)

Exponentiation

Goal: one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)
```

Linear time:

- Doubling the input **doubles** the time
- 1024x the input takes 1024x as much time

```
def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
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def square(x):
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```

Logarithmic time:

- Doubling the input **increases** the time by a constant C
- 1024x the input increases the time by only 10 times C

Orders of Growth

Quadratic Time

Functions that process all pairs of values in a sequence of length n take quadratic time

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def overlap(a, b):
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    for item in a:
        for other in b:
            if item == other:
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    return count

overlap([3, 5, 7, 6], [4, 5, 6, 5])
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Tree-recursive functions can take exponential time

```
def fib(n):
    if n == 0:
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        return fib(n-2) + fib(n-1)
```

http://en.wikipedia.org/wiki/File:Fibonacci.jpg

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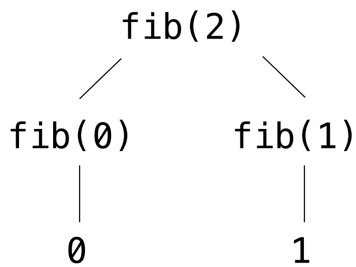

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```
              fib(3)
             /      \
        fib(1)        fib(2)
          |          /      \
          1     fib(0)      fib(1)
                  |           |
                  0           1
```

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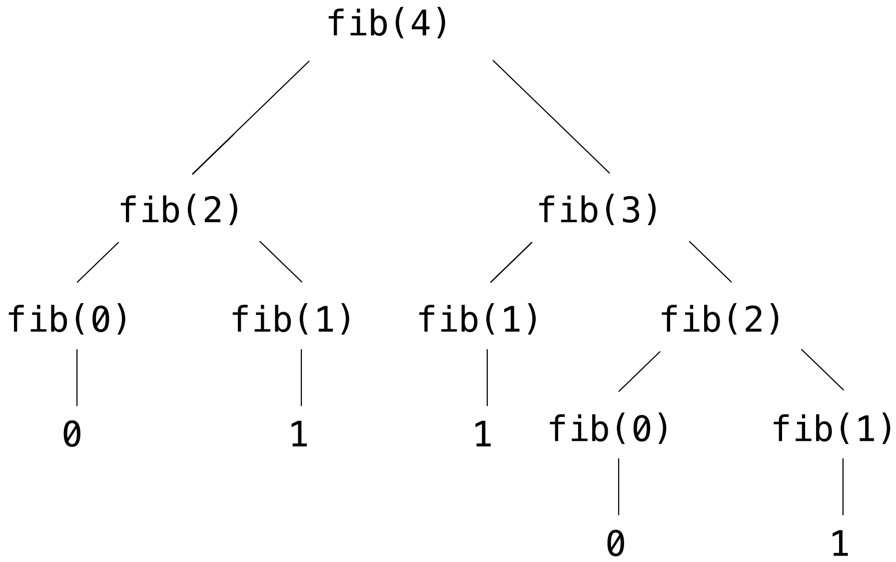

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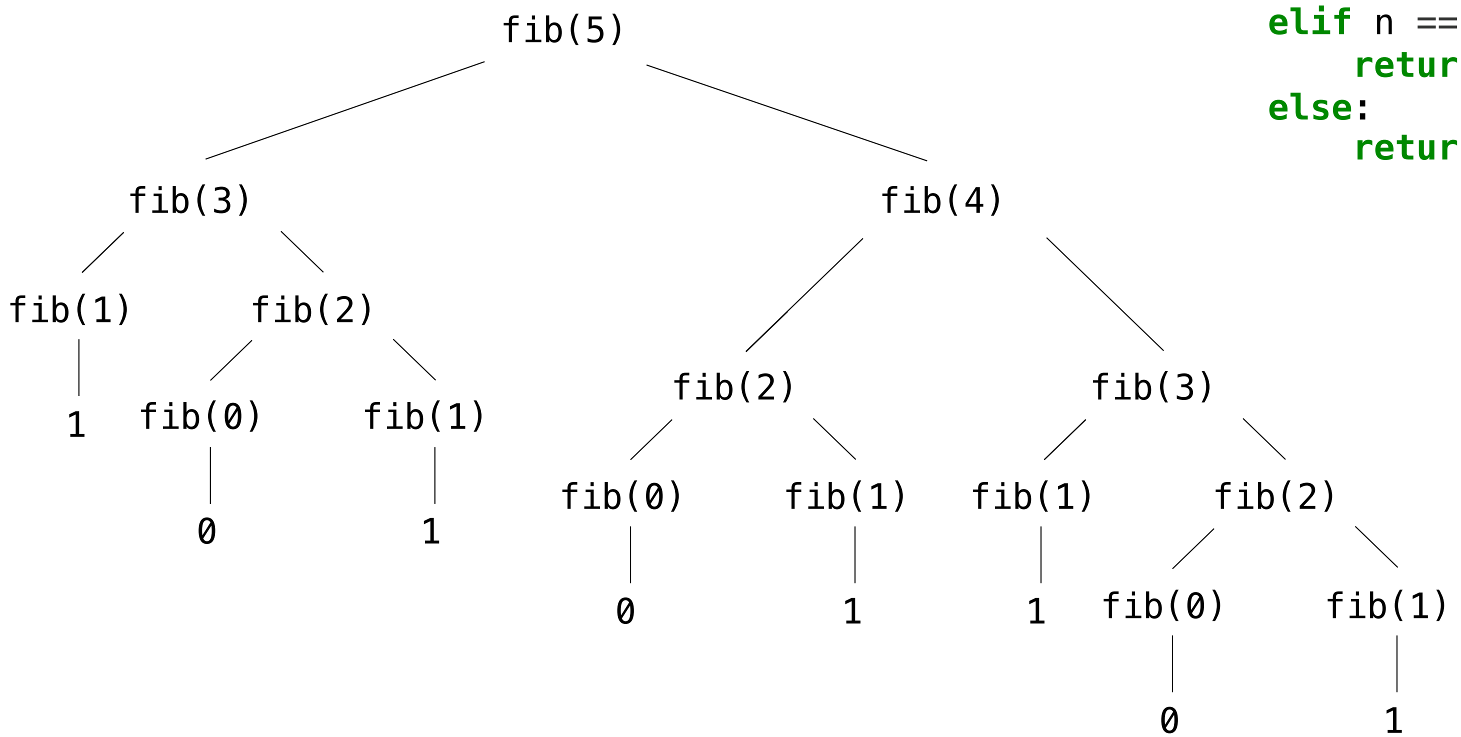

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Common Orders of Growth

Exponential growth. E.g., recursive `fib`

Quadratic growth. E.g., `overlap`

Linear growth. E.g., slow `exp`

Logarithmic growth. E.g., `exp_fast`

Constant growth. Increasing *n* doesn't affect time

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$$a \cdot b^{n+1} = (a \cdot b^n) \cdot b$$

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Big Theta and Big O Notation for Orders of Growth

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Space

Space and Environments

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```
Which environment frames do we need to keep during evaluation?
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(Demo)

pythontutor.com/
composingprograms.html#code=def%20fib%28n%29%3A%0A%20%20%20%20if%20n%20%3D%3D%200%20or%20n%20%3D%3D%201%3A%0A%20%20%20%20%20%20%20%20return%20n%0A%20%20%20%20else%3A%0A%20%20%20%20%20%20%20%20return%20fib%28n-2%29%20%2B%20fib%28n-1%29%0A%20%20%20%20%20%20%20%20%0Afib%286%29&mode=display&
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Fibonacci Space Consumption

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```
fib(5)
```

Fibonacci Space Consumption

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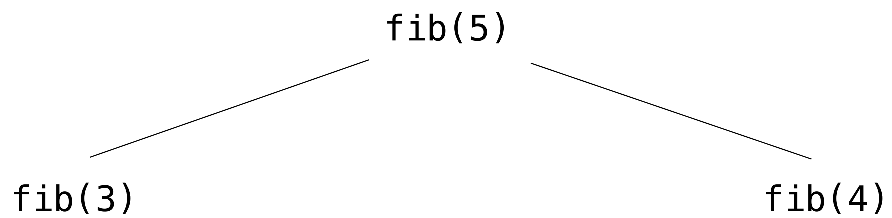
fib(3)

Fibonacci Space Consumption



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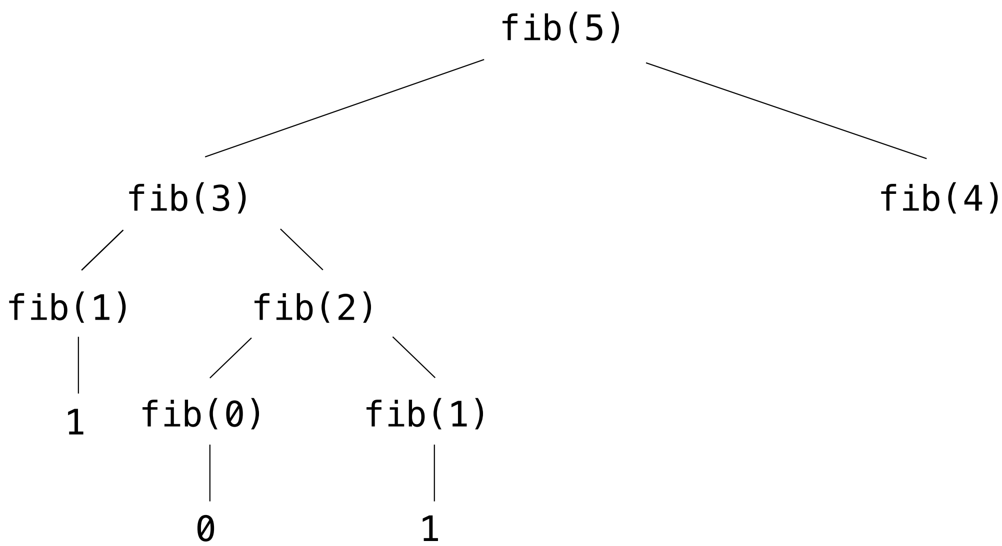

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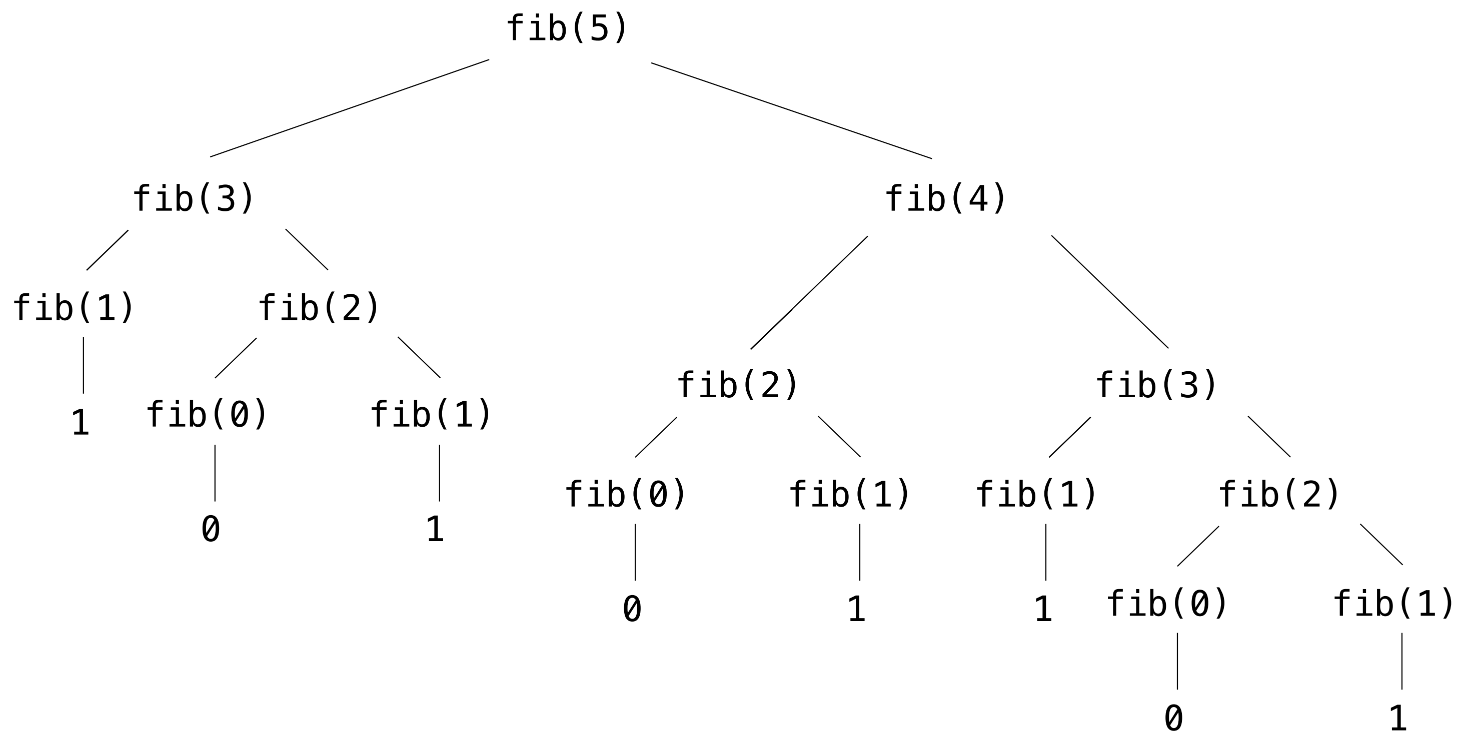

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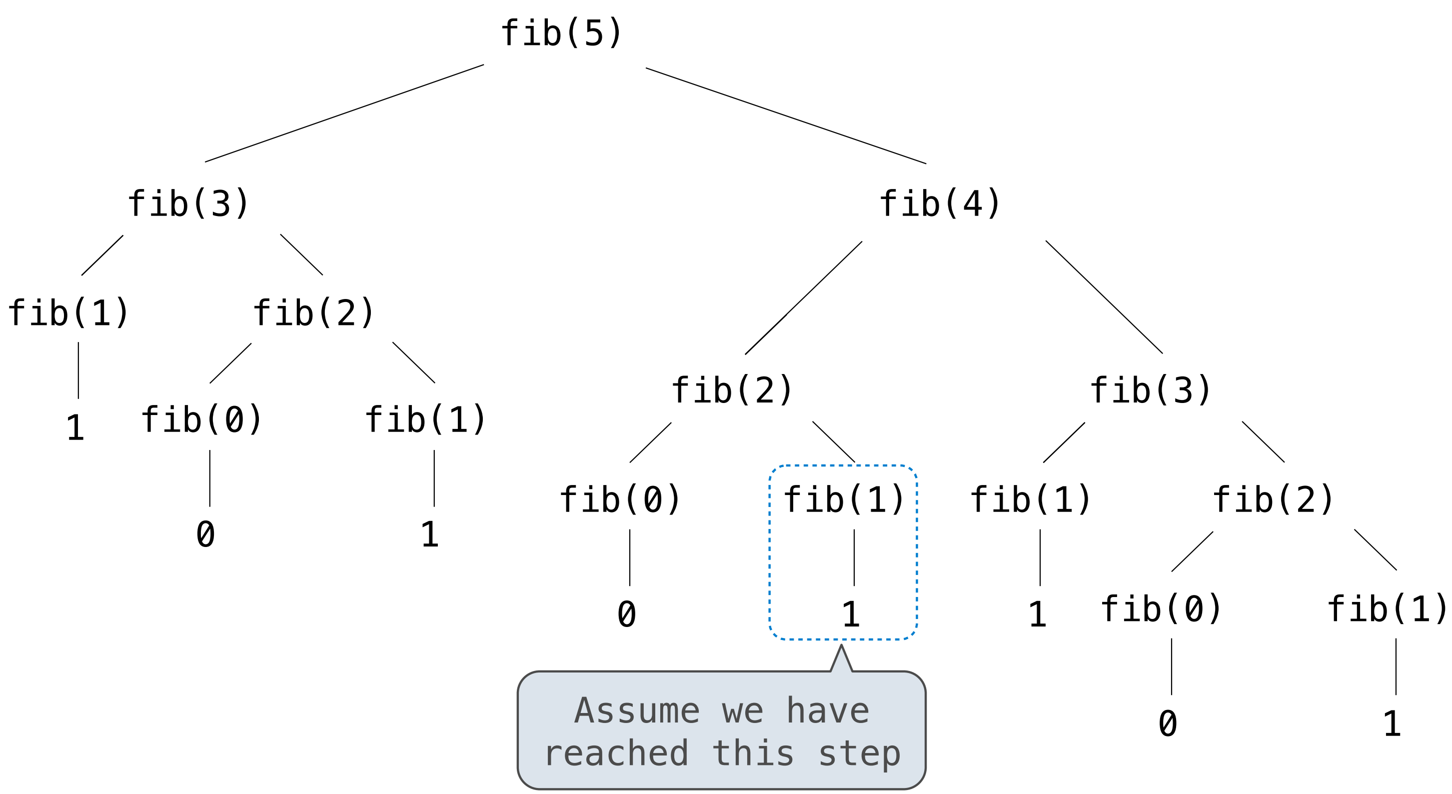

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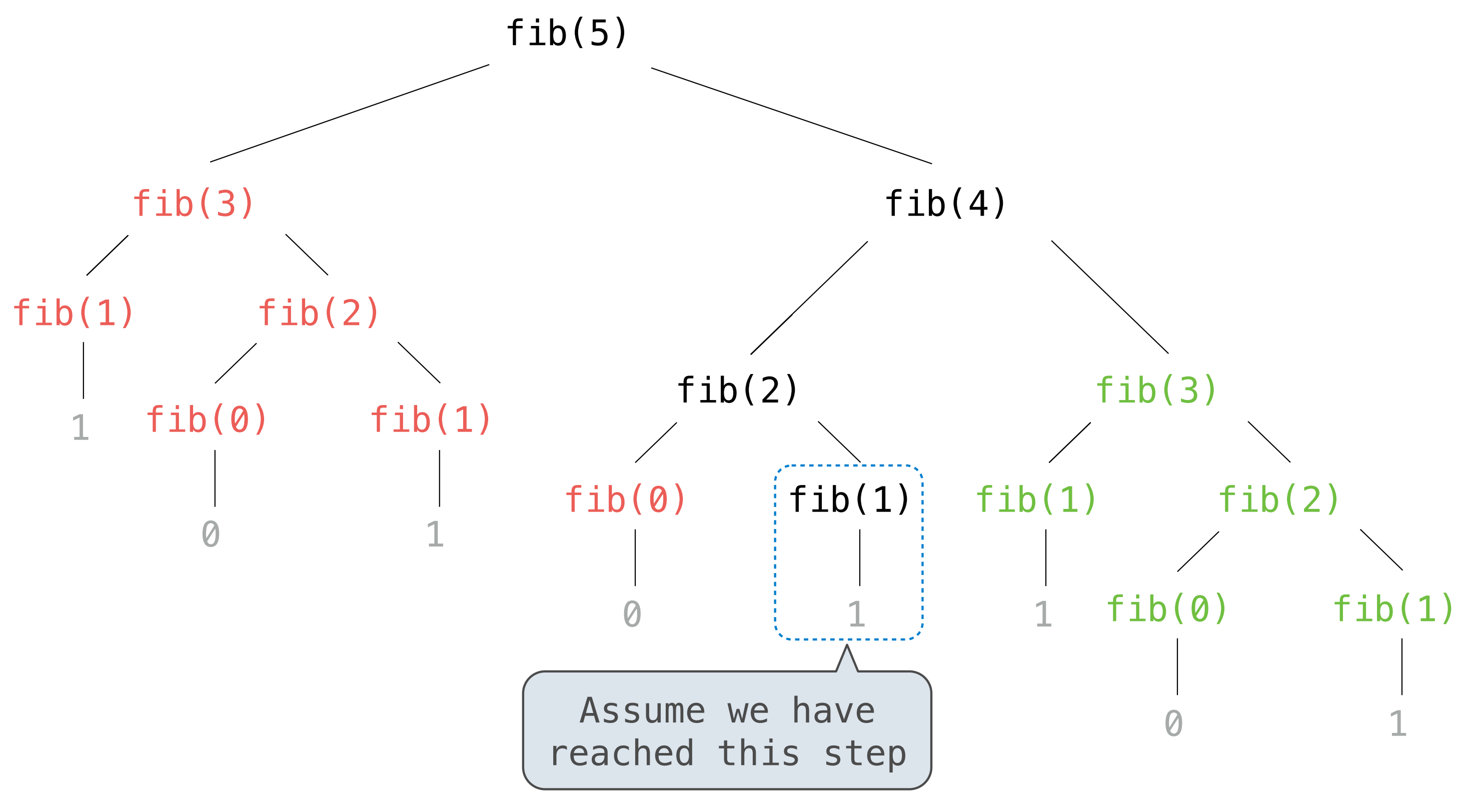

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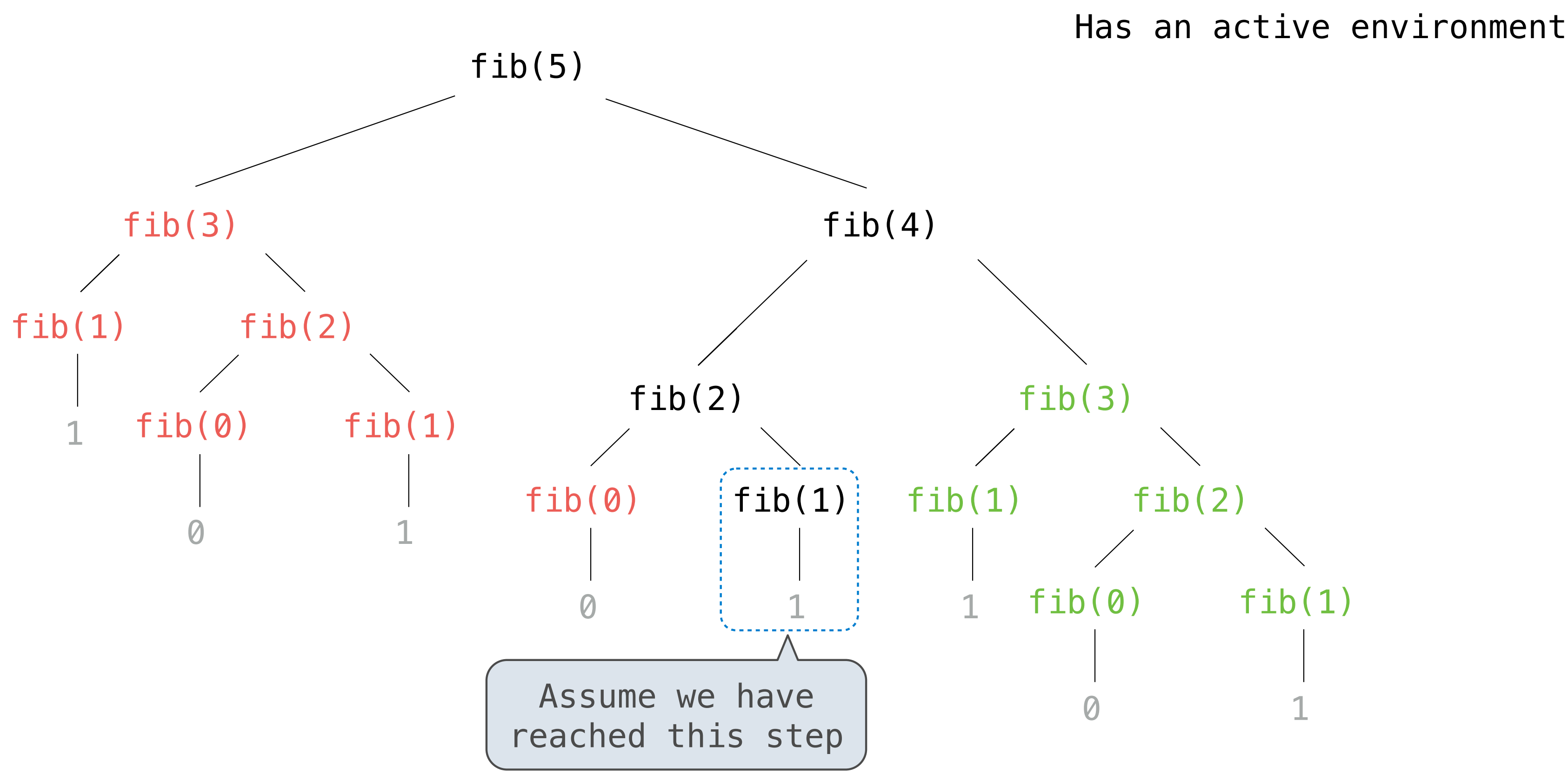

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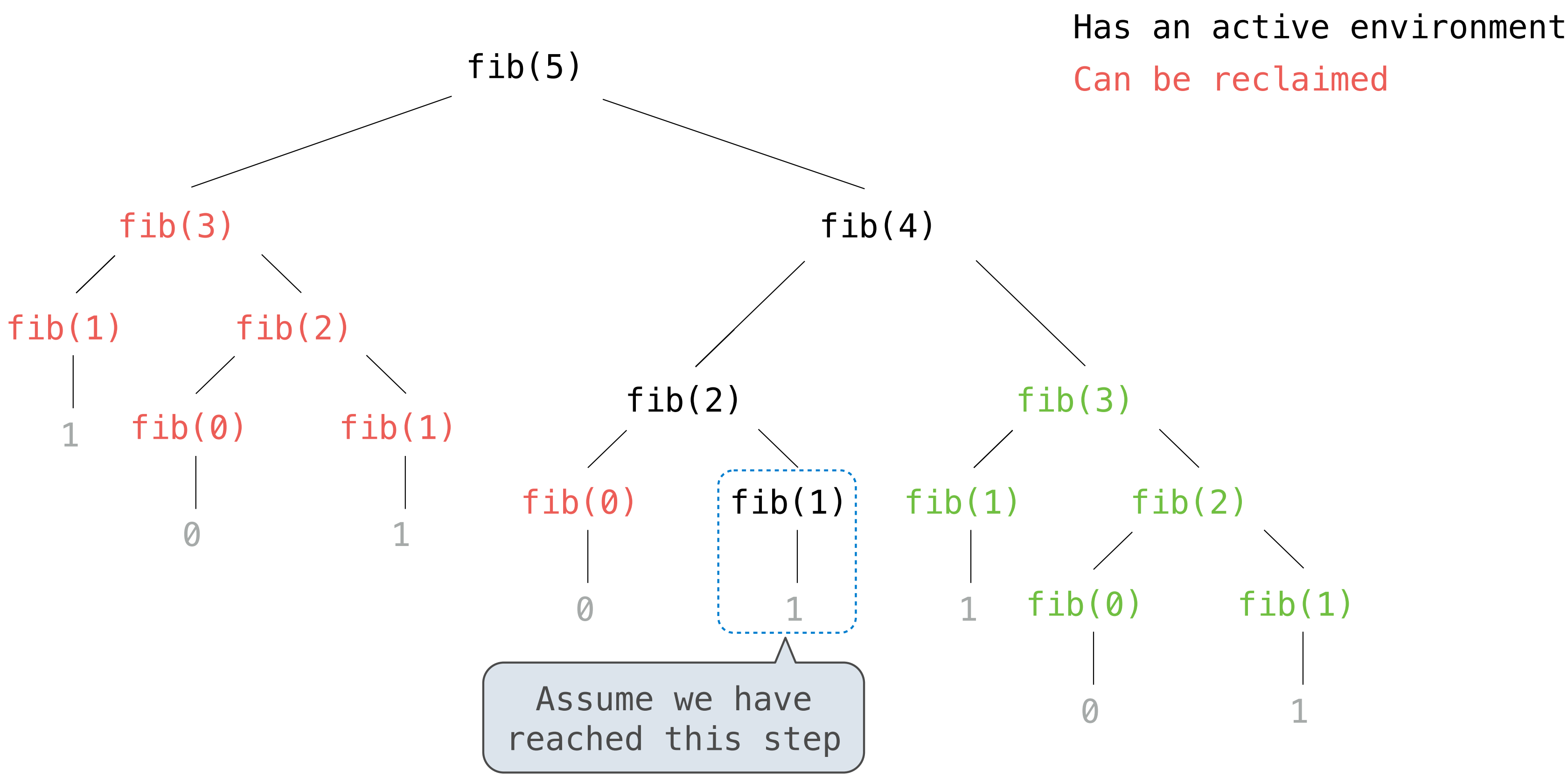

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